CANONICAL SYMBOLIC DYNAMICS FOR
ONE-DIMENSIONAL GENERALIZED SOLENOIDS

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ABSTRACT. We define canonical subshift of finite type cover for Williams' 1-dimensional generalized solenoids, and use resulting invariants to distinguish some closely related solenoids.

1. INTRODUCTION

R. F. Williams has developed a theory of expanding attractors for a dynamical system ([14, 15, 16]). These can be modeled as shift maps of generalized n-solenoids which are defined as inverse limits of immersions of n-dimensional branched manifolds satisfying certain axioms.

In this paper, we produce canonical shift of finite type (SFT) covers of Williams' 1-solenoids in the following sense: Let $\overline{X}$ be a 1-solenoid, $\overline{f}$ a shift map on $\overline{X}$, and $\overline{\mathcal{O}}$ a union of finitely many periodic orbits of $\overline{f}$. We give an algorithm which takes the input $\{\overline{X}, \overline{f}, \overline{\mathcal{O}}\}$ and produces as output a mixing SFT $\Sigma_{\mathcal{O}}$ with shift map $\sigma_{\mathcal{O}}$ and a semiconjugacy $p_{\mathcal{O}} \colon \Sigma_{\mathcal{O}} \to \overline{X}$ (that is, $p_{\mathcal{O}}$ is a continuous surjection and $p_{\mathcal{O}} \circ \sigma_{\mathcal{O}} = \overline{f} \circ p_{\mathcal{O}}$). Then we prove that if there is a conjugacy ϕ of 1-solenoids $(\overline{X}, \overline{f})$ and $(\overline{X'}, \overline{f'})$ such that ϕ sends $\overline{\mathcal{O}}$ to $\overline{\mathcal{O'}}$, then there is a unique conjugacy $\tilde{\phi}$ such that $\phi \circ p_{\mathcal{O}} = p_{\mathcal{O'}} \circ \tilde{\phi}$. These covers can be exploited to give nontrivial computable invariants distinguishing closely related solenoid maps.

To our knowledge, we are giving the first construction of canonical SFT covers for a class of nonzero dimensional systems. The canonicalness requires the dependence on $\mathcal{O}$, and perhaps this is why it was not noticed earlier. However, there have been other works ([1, 11]) achieving some specialness or invariant for SFT covers of some systems, and there were earlier constructions of canonical covers for some systems. Krieger ([8]) gave canonical SFT covers of sofic systems (which are zero dimensional), and Fried ([4, 7]) more generally offered canonical coordinate (CC) covers of finitely presented (FP) systems. These covers are built from sets of possible pasts and futures. The Krieger-Fried covers make the step from FP to CC. The 1-solenoids are already CC, and the covers we produce are SFTs. We raise the question, can our relatively simple one-dimensional construction be generalized in some inductive way to produce canonical symbolic dynamics for higher dimensional generalized solenoids?

Apart from the matter of canonical symbolic dynamics, we mention renewed interest in Williams' systems and related systems on account of connections with ordered group invariants ([3, 12, 18]) and substitutions and tilings ([2, 6]).

We study the 1-solenoids as purely topological systems. For this we give some defining topological axioms closely related to Williams' axioms. A 1-solenoid of Williams becomes one of our 1-solenoids by ignoring the differentiable structure. Conversely, every topologically defined 1-solenoid can be given a differentiable structure which makes it a 1-solenoid in the sense of Williams. However, the essential aspects of the situation are not differentiable but topological, and to clarify this we give the purely topological development.

The outline of the paper is as follows. In section 2, following Williams ([14, 15]) rather closely, we give axioms for our systems and prove some basic facts about them. We also recall the construction of an SFT cover from a graph presentation. In section 3, we recall Williams' definition of shift equivalence and show that every topological conjugacy of branched 1-solenoids is induced by a shift equivalence of their graph presentations. (A 'branched' solenoid is a solenoid derived from a presentation which need not satisfy Williams' Flattening Axiom, so this is a slight generalization of Williams' work.) We also establish a key observation: If the shift equivalence is given by graph maps (maps sending vertices to vertices), then the conjugacy lifts uniquely to a conjugacy of the SFT covers derived from the graph presentations. In section 4, given $\overline{\mathcal{O}}$, we give a graph algorithm for a new graphical presentation $(X_{\mathcal{O}}, f_{\mathcal{O}})$ of the solenoid system. One consequence of this construction is that every 1-solenoid with a fixed point has an elementary presentation in the sense of Williams, so this extends Williams' classification result ([15, §7]) to all 1-solenoids with fixed points. In Section 5, we use the previous results to produce the canonical SFT covers, and use them to distinguish the pair of systems considered by Williams and Ustinov ([13, 15]) by computing Bowen-Franks groups of certain attached canonical SFT covers. In Appendix A, we show that our canonical SFT covers are not canonical as one-sided SFTs, despite the one-sided aspects of the construction. In Appendix B, we show that our topological 1-solenoids can be given a differentiable structure making them differentiable immersions in the sense of Williams.

2. Markov maps and their SFT covers

In the style of Williams ([14, 15]), we will define several axioms which might be satisfied by a continuous self-map of a directed graph. Let X be a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and $f\colon X \to X$ a continuous map. Axioms 0-3 and 5 correspond to Williams' Axioms 0-2, 3°, and 4 in [15].

Axiom 0. (*Indecomposability*) (X, f) is indecomposable.

Axiom 1. (*Nonwandering*) All points of X are nonwandering under f.

Axiom 2. (*Flattening*) There is $k \geq 1$ such that for all $x \in X$ there is an open neighborhood U of x such that $f^k(U)$ is homeomorphic to $(-\epsilon, \epsilon)$.

Axiom 3. (*Expansion*) There are a metric d compatible with the topology and positive constants C and λ with $\lambda > 1$ such that for all $n > 0$ and all points x, y on a common edge of X, if f^n maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \geq C\lambda^n d(x, y)$.

Axiom 4. (*Nonfolding*) $f^n|_{X-\mathcal{V}}$ is locally one-to-one for every positive integer n.

Axiom 5. (*Markov*) $f(\mathcal{V}) \subseteq \mathcal{V}$.

Remarks 2.1. (1) Axiom 0 means that X cannot be split into two nonempty, closed, f-invariant subsets ([15, §1]).

(2) We can define (without derivatives) an arclength from the assumed metric d as follows. Suppose $\gamma \colon [0,1] \to X$ where γ is continuous and locally one-one. Let $t_1, \ldots, t_{n-1}$ be all elements of $(0,1)$ which γ maps to vertices of X. Define the length of γ as $\sum_{i=1}^{n} d(x_{i-1}, x_i)$. With this definition, we can say that the Expansion Axiom means that there exists a metric compatible with the topology of X such that there are constants $C > 0$ and $\lambda > 1$ such that f^n increases arclength by a factor of at least $C\lambda^n$ (this was one formulation of Williams [14, 15], except that his arclength tacitly was defined as usual with derivatives). Also, if we define a path metric d' by setting $d'(x,y)$ to be the length of the shortest path from x to y, then d' is another metric compatible with the topology and still satisfies the Expansion Axiom.

Standing Assumption. In this paper, we always assume that (X, f) satisfies Axioms 0 and 1.

For a given directed graph X with a continuous map $f \colon X \to X$, let $\overline{X}$ be the inverse limit space

$$\overline{X} = X \xleftarrow{\;f\;} X \xleftarrow{\;f\;} \cdots = \big\{ (x_0, x_1, x_2, \ldots) \in \prod_{0}^{\infty} X \mid f(x_{n+1}) = x_n \big\},$$

and $\overline{f} \colon \overline{X} \to \overline{X}$ the induced homeomorphism defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (f(x_0), f(x_1), f(x_2), \ldots) = (f(x_0), x_0, x_1, \ldots).$$

Let Y be a topological space and $g \colon Y \to Y$ a homeomorphism. We call Y a **1-dimensional generalized solenoid** or **1-solenoid** and g a **solenoid map** if there exist a directed graph X and a graph map $f \colon X \to X$ such that (X, f) satisfies all six Axioms and $(\overline{X}, \overline{f})$ is topologically conjugate to (Y, g). We say that (X, f) is a **presentation** of (Y, g). If (X, f) satisfies all Axioms except possibly the Flattening Axiom, then we call Y a **branched solenoid**.

Remarks 2.2. (1) Williams defined an n-dimensional generalized solenoid $\overline{X}$ and a solenoid map $\overline{f}$ as the inverse limit of a system (X, f) satisfying Axioms 0-3 where X is a *differentiable n-dimensional branched manifold* and $f \colon X \to X$ is an *immersion* ([14, 15, 16]). We generalize his systems in the topological category for the 1-dimensional case. As a topological system, every 1-solenoid of Williams is a 1-solenoid by our topological definition. We will see the relation between Williams' definition and the topological definition in Appendix B.

(2) The Nonfolding Axiom is the topological condition we use in place of the differentiable immersion condition.

(3) If (X, f) satisfies Axioms 0-4, then there is a presentation (X', f') satisfying Axioms 0-5 such that $(\overline{X}, \overline{f})$ is topologically conjugate to $(\overline{X'}, \overline{f'})$ ([14, Proposition 3.5]). Williams proved [14, 3.5] assuming the immersion condition, but his proof goes through with our Axioms 0-4.

Example 2.3. Let X be the unit circle on the complex plane. Suppose that 1 and -1 are the vertices of X, and that the upper half circle e_1 and the lower half circle e_2 with counterclockwise direction are the edges of X. Define $f, g : X \to X$ by

$$f : z \mapsto z^2 \text{ and } g : z \mapsto \begin{cases} z^2 & \text{if } z \in e_1 \\ z^{-2} & \text{if } z \in e_2. \end{cases}$$

Then (X, f) satisfies all six Axioms, and (X, g) satisfies all Axioms except the Nonfolding Axiom. For (X, g), g^2 is not locally one-to-one at $\exp(\frac{\pi}{2}i) \in S^1$.

Notation 2.4. Suppose that (X, f) satisfies the Nonfolding Axiom and the Markov Axiom, $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the edge set of X with a given direction, and k is a positive integer. For each edge $e_i \in \mathcal{E}$, we can *give e_i the partition* $\{I_{i,j}^{(k)}\}$, $1 \leq j \leq j(i, k)$, *for f^k* such that

  (1) the initial point of $I_{i,1}^{(k)}$ is the initial point of e_i,
  (2) the terminal point of $I_{i,j}^{(k)}$ is the initial point of $I_{i,j+1}^{(k)}$ for $1 \leq j < j(i, k)$,
  (3) the terminal point of $I_{i,j(i,k)}^{(k)}$ is the terminal point of e_i,
  (4) $f^k|_{\mathrm{Int}\, I_{i,j}^{(k)}}$ is injective, and
  (5) $f^k(I_{i,j}^{(k)}) = e_{i,j}^{(k)\,s(i,j,k)}$ where $e_{i,j}^{(k)} \in \mathcal{E}$, $s(i, j, k) = 1$ if the direction of $f^k(I_{i,j}^{(k)})$ agree with that of $e_{i,j}^{(k)}$, and $s(i, j, k) = -1$ if the direction of $f^k(I_{i,j}^{(k)})$ is reverse to that of $e_{i,j}^{(k)}$.

Remarks 2.5.   (1) By the Nonfolding Axiom, if $f^k(I_{i,j}^{(k)}) = e_{i,j}^{\pm 1}$ for $1 \leq j < j(i, k)$, then $f^k(I_{i,j+1}^{(k)}) = e_{i,j+1}$ cannot be $e_{i,j}^{\mp 1}$.
  (2) If (X, f) satisfies all six axioms, then there is a positive integer m such that, for every vertex v of X and every integer $k \geq m$, there exist at most two edges $e_{v,k,1}$ and $e_{v,k,2}$ such that, for every $I \in \{I_{i,j}^{(k)} \mid v \in f^k(I_{i,j}^{(k)})\}$, $f^k(I) = e_{v,k,l}^{\pm 1}$, $l = 1$ or 2.

Definition 2.6. Suppose that (X, f) is a presentation of a branched solenoid, that is, (X, f) satisfies all Axioms except possibly the Flattening Axiom, and $\mathcal{E}$ is the edge set of X. Then each edge $e_i \in \mathcal{E}$ has the partition $\{I_{i,j}^{(1)}\}$ for f, and we can define an induced map $\tilde{f} \colon \mathcal{E} \to \mathcal{E}^*$ by

$$\tilde{f} \colon e_i \mapsto e_{i,1}^{s(i,1)} e_{i,2}^{s(i,2)} \cdots e_{i,j(i)}^{s(i,j(i))}$$

where $e_{i,j}^{s(i,j)} = f(I_{i,j}^{(1)})$ for $1 \leq j \leq j(i)$. We call $\tilde{f}$ the **substitution rule** or the **wrapping rule** associated to f.

Examples 2.7. Let (X, f) and (X, g) be given in Examples 2.3. Then $\tilde{f}, \tilde{g} \colon \mathcal{E}_X \to \mathcal{E}_X^*$ are given by

$$\tilde{f} \colon e_1 \mapsto e_1 e_2, \ e_2 \mapsto e_1 e_2, \ \text{and} \ \tilde{g} \colon e_1 \mapsto e_1 e_2, \ e_2 \mapsto e_2^{-1} e_1^{-1}.$$

To establish some notation, we give Figure 1 to represent the presentations (X, f) and (X, g) with the wrapping rules $\tilde{f}$ and $\tilde{g}$, respectively.

Similarly, if (Y, h) is given by Figure 2, then the wrapping rule $\tilde{h} \colon \mathcal{E}_Y \to \mathcal{E}_Y^*$ is given by

$$a \mapsto cabd, \quad b \mapsto dc, \quad c \mapsto ab, \quad d \mapsto abc.$$

Note that the two vertices q, r of Y have h-period 2. If U_q and U_r are sufficiently small neighborhoods of q and r, respectively, then $h^2(U_q)$ and $h(U_r)$ are intervals. So (Y, h) satisfies the Flattening Axiom.
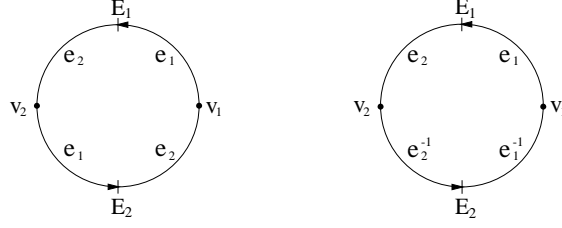
FIGURE 1. (X, f) and (X, g) with the wrapping rules $\tilde{f}$ and $\tilde{g}$, respectively.
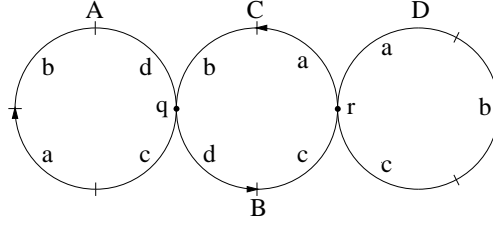
FIGURE 2. (Y, h) with wrapping rule $\tilde{h}$.

Lemma 2.8. *Suppose that (X, f) satisfies Axioms 4 and 5. Then there is a positive integer l such that, for each edge e_i of X and every positive integer m, if $\{I_{i,1}^{(l)}, \ldots, I_{i,j(i,l)}^{(l)}\}$ and $\{I_{i,1}^{(lm)}, \ldots, I_{i,j(i,lm)}^{(lm)}\}$ are partitions of e_i for f^l and f^{lm}, respectively, then $f^l(I_{i,1}^{(l)}) = f^{lm}(I_{i,1}^{(lm)})$ and $f^l(I_{i,j(i,l)}^{(l)}) = f^{lm}(I_{i,j(i,lm)}^{(lm)})$.*

Proof. Since $\mathcal{V}$ is a finite set and $f(\mathcal{V}) \subset \mathcal{V}$, every vertex of X is eventually periodic, and there is a positive integer l_1 such that $f^{l_1}(v) = f^{l_1 m}(v)$ for every $v \in \mathcal{V}$ and every positive integer m.

If e_i is an edge of X beginning at v, and $\{I_{i,1}^{(l_1)}, \ldots, I_{i,j(i,l_1)}^{(l_1)}\}$ is the partition of e_i for f^{l_1}, then $f^{l_1}(I_{i,1}^{(l_1)}) = e_j^s$ such that $f^{l_1}(v) \in e_j$. Since $\mathcal{E}$ is a finite set and $f^{l_1}(v)$ is a fixed point of f^{l_1}, there is a positive integer l_2 such that, for every positive integer m, if e_j has partitions $\{I_{j,1}^{(l_2)}, \ldots\}$ for f^{l_2} and $\{I_{j,1}^{(l_2 m)}, \ldots\}$ for $f^{l_2 m}$, then $f^{l_2}(I_{j,1}^{(l_2)}) = f^{l_2 m}(I_{j,1}^{(l_2 m)})$. This shows that $f^{l_1 l_2}(I_{i,1}^{(l_1 l_2)}) = f^{l_1 l_2 m}(I_{i,1}^{(l_1 l_2 m)})$ for every positive integer m. By the same argument, we can choose a positive integer l' for e_i such that $f^{l'}(I_{i,j(i,l')}^{(l')}) = f^{l' m}(I_{i,j(i,l' m)}^{(l' m)})$.

Let ℓ_i be the least common multiple of $l_1 l_2$ and l' for each edge e_i, and l the least common multiple of these ℓ_i's. Then we have $f^l(I_{i,1}^{(l)}) = f^{lm}(I_{i,1}^{(lm)})$ and $f^l(I_{i,j(i,l)}^{(l)}) = f^{lm}(I_{i,j(i,lm)}^{(lm)})$ for every positive integer m. $\qquad\square$

Lemma 2.9. *Suppose that (X, f) satisfies Axioms $3, 4$ and 5. Then there exist a positive integer l and $\epsilon > 0$ such that, for all $x, y \in X$, $f^l(x) \neq f^l(y)$ implies that there is a nonnegative integer K such that $d(f^K(x), f^K(y)) \geq \epsilon$.*

Proof. For convenience, we will take the metric d on X so that $d(x, y)$ is the length of the shortest path between x and y, as explained in Remarks 2.1.

Let l be the integer given in Lemma 2.8. So each edge e_i has the partition $\{I_{i,j}^{(l)}\}$ for $f^l \colon X \to X$ as in Notation 2.4. Without loss of generality, we suppose each

$j(i, l) \geq 3$. Let $\mathcal{P}$ be the collection of the intervals $I_{i,j}^{(l)}$. First choose $\epsilon_1 > 0$ so small that

  (i) each interval $I_{i,j}^{(l)}$ has length larger than $2\epsilon_1 > 0$,
  (ii) if $f^l(x) = a \in I_{i,1}^{(l)}$, v is the initial point of e_i, and $d(v, a) < \epsilon_1$, then $f^l(a) \in I_{i,1}^{(l)}$, and
  (iii) if $f^l(y) = b \in I_{i,j(i,l)}^{(l)}$, w is the terminal point of e_i, and $d(w, b) < \epsilon_1$, then $f^l(b) \in I_{i,j(i,l)}^{(l)}$.

Then choose ϵ such that $0 < \epsilon < \epsilon_1$ and for every x in the compact set $\displaystyle\bigcup_{\substack{i \\ 1 < j < j(i,l)}} I_{i,j}^{(l)}$

and every $y \in X$, if $0 < d(x, y) < \epsilon$, then

  (iv) $0 < d(f^l(x), f^l(y)) < \epsilon_1$ and
  (v) the interval $[f^l(x), f^l(y)]$ contains at most one vertex.

Note that $f^l(x) \neq f^l(y)$ comes from the Nonfolding Axiom.

If $f^l(x) = a \neq b = f^l(y)$ and $d(a, b) < \epsilon_1$, then a and b lie on the same or adjacent elements of $\mathcal{P}$. So we have two cases:

  (1) a and b are end points of an interval of length less than ϵ_1 containing a vertex v, or
  (2) the interval $[a, b]$ of length less than ϵ_1 does not contain any vertex of X.

For case (1), by the condition (i), $d(a, b) < \epsilon_1$ implies that a and b are contained in two different intervals among $I_{i,1}^{(l)}$, $I_{i,j(i,l)}^{(l)}$, $I_{n,1}^{(l)}$, and $I_{n,j(n,l)}^{(l)}$. For brevity, let's assume $a \in I_{i,1}$ and $b \in I_{n,1}$. Then by the condition (ii), $f^l(a) \in I_{i,1}^{(l)}$ and $f^l(b) \in I_{n,1}^{(l)}$. If v is the vertex of X contained in $I_{i,1}^{(l)} \cap I_{n,1}^{(l)}$, then f^l maps $[v, a]$ into $I_{i,1}$. So

$$d(f^l(a), v) \geq c\lambda^l \cdot d(a, v)$$

where c and λ are the expansion constants. Similarly $d(f^l(b), v) \geq c\lambda^l \cdot d(b, v)$. Let k be the smallest positive integer such that $d(v, f^{lk}(a)) \geq \epsilon_1$ or $d(v, f^{lk}(b)) \geq \epsilon_1$. Then by induction using (ii) and (iii), we have for $0 < s \leq k$ that f^{sl} sends $[v, a]$ injectively into $I_{i,1}$ and $[v, b]$ injectively into $I_{n,1}$. Therefore we have $d(f^{kl}(a), f^{kl}(b)) \geq \epsilon_1 > \epsilon$.

For case (2), let k be the smallest positive integer such that $[f^{kl}(a), f^{kl}(b)]$ contains a vertex. It follows from the Nonfolding Axiom that $f^{(k-1)l}(a) \neq f^{(k-1)l}(b)$. If there exists i, $0 < i < k$, such that $d(f^{il}(a), f^{il}(b)) > \epsilon$, then we are done, so suppose not. Then $f^{kl}(a)$ and $f^{kl}(b)$ are endpoints of an interval of length less than ϵ_1 containing a vertex. Hence we may conclude the proof by applying the argument of case (1). $\qquad\square$

Definition 2.10 ([10, §3.5]). A homeomorphism h on a metric space Y is **expansive** if there is an $\epsilon > 0$ such that, for all $x \neq y \in Y$, there is an integer n such that

$$d(h^n(x), h^n(y)) \geq \epsilon.$$

For a solenoid $\overline{X}$ presented by (X, f), define a metric $\bar{d}$ on $\overline{X}$ by

$$\bar{d}(x, y) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$$

where $x = (x_0, x_1, \dots), y = (y_0, y_1, \dots) \in \overline{X}$, and d is a metric on X compatible with the topology of X such that $f \colon X \to X$ is an expansion with respect to d as in Remarks 2.1.(2).

Proposition 2.11. *If (X, f) satisfies Axioms $3, 4$ and 5, then $\overline{f} \colon \overline{X} \to \overline{X}$ is expansive.*

Proof. For a pair of points $x = (x_0, x_1, \dots) \neq y = (y_0, y_1, \dots) \in \overline{X}$, there is a nonnegative integer N such that $x_n \neq y_n$ for all $n \geq N$.

Let l and $\epsilon > 0$ be given in Lemma 2.9. Then $x_{N+l} \neq y_{N+l}$ and $f^l(x_{N+l}) \neq f^l(y_{N+l})$ imply that there exists $K \geq 0$ such that $d\big(f^K(x_{N+l}), f^K(y_{N+l})\big) \geq \epsilon$ by Lemma 2.9. Therefore we have

$$\bar{d}\big(\overline{f}^{K-N-l}(x), \overline{f}^{K-N-l}(y)\big) = d\big(f^K(x_{N+l}), f^K(y_{N+l})\big) + \sum_{i > K-N-l} \frac{d\big(x_i, y_i\big)}{2^{N+l-K+i}} > \epsilon,$$

and this proves that $\overline{f} \colon \overline{X} \to \overline{X}$ is expansive. □

Example 2.12. We need the Nonfolding Axiom in Proposition 2.11.

For $f, g \colon S^1 \to S^1$ in Examples 2.3, it follows from Proposition 2.11 that $\overline{f}$ is expansive. For $\overline{g}$ with given $\epsilon > 0$ and $x = (x_0, x_1, \dots), y = (y_0, y_1, \dots) \in (\overline{S^1}, \overline{g})$, let

$$x_n = \exp(\frac{\pi}{2^n}i + \frac{1}{2^{n+2}}\epsilon i) \text{ and } y_n = \exp(\frac{\pi}{2^n}i - \frac{1}{2^{n+2}}\epsilon i).$$

Then, for a natural Riemannian metric d on S^1,

$$\bar{d}(\overline{g}^k(x), \overline{g}^k(y)) = \frac{1}{2^{|k|}}d(x, y) = \frac{1}{2^{|k|}}\epsilon$$

for every integer k, and $\overline{g}$ is not expansive.

SFT covers. We will review the standard construction of a shift of finite type (SFT) cover for 1-dimensional branched solenoids.

Suppose that (X, f) is a presentation of a branched solenoid, and $\mathcal{E} = \{e_1, \dots, e_n\}$ is the edge set of X. Let $\{I_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq j(i)\}$ be the partition of $\mathcal{E}$ for f, and $\tilde{f} \colon \mathcal{E} \to \mathcal{E}^*$ the wrapping rule associated to f given by

$$(1) \qquad \tilde{f} \colon e_i = I_{i,1} \cdots I_{i,j(i)} \mapsto e_{i,1}^{s(i,1)} e_{i,2}^{s(i,2)} \cdots e_{i,j(i)}^{s(i,j(i))}$$

where $e_{i,j}^{s(i,j)} = f(I_{i,j})$ and $s(i, j) = \pm 1$ denotes the direction. The *adjacency matrix* M of $(\mathcal{E}, \tilde{f})$ is given by

$$M(i, j) = \#\{I_{i,l} \mid f(I_{i,l}) = e_j^{\pm 1}\}.$$

We may view M as the adjacency matrix of a directed graph whose vertex set is $\mathcal{E}$ and whose edge set is $\mathcal{A} = \{I_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq j(i)\}$, the partition of $\mathcal{E}$ for f, where outgoing edges from the 'vertex' e_i are named $I_{i,1}, \dots, I_{i,j(i)}$.

Now we can give $(\overline{X}, \overline{f})$ a two-sided SFT cover (Σ_X, σ_X) defined from the *alphabet* $\mathcal{A}$ and the adjacency matrix M_X. The shift space Σ_X is the subset of $\mathcal{A}^{\mathbb{Z}}$ whose forbidden blocks are $\{I_{i,j}I_{k,l} \mid I_{k,l} \nsubseteq f(I_{i,j}) = e_{i,j}^{s(i,j)}\}$ from the formula (1). Therefore Σ_X is a 1-step subshift of finite type, and a word $w = I_{a(0)}I_{a(1)} \cdots I_{a(l)}$ is *allowed* in Σ_X if and only if $\bigcap_{j=0}^{l} f^{-j}(I_{a(j)})$ has nonempty interior.

For each point $w = \cdots I_{a(-1)} I_{a(0)} I_{a(1)} \cdots \in \Sigma_X$ and the canonical projection map onto the zeroth coordinate $\pi\colon \overline{X} \to X$, there is a unique corresponding point

$$x_w = (x_0, x_1, \dots) = \bigcap_{j=-\infty}^{\infty} \overline{f}^{-j}\big(\pi^{-1}(I_{a(j)})\big) \in \overline{X}$$

such that $x_i \in I_{a(-i)}$ and $f^i(x_0) \in I_{a(i)}$ for $i \geq 0$. Hence there is a well-defined semiconjugacy $p\colon \Sigma_X \to \overline{X}$ defined by $w \mapsto x_w$. It is not difficult to check $\overline{f} \circ p = p \circ \sigma_X$ ([9, §6.5]).

Proposition 2.13 ([4, §3.D]). *Suppose that (X, f) satisfies all six Axioms except possibly the Flattening Axiom. Let p and Σ_X be as above. Then*

(1) $p\colon \Sigma_X \to \overline{X}$ *is finite-to-one.*

(2) p *is one-to-one on* $\Sigma_X \setminus \bigcup_{m=0}^{\infty} p^{-1} \circ \pi^{-1} \circ f^{-m-1}(\mathcal{V})$ *where* $\pi\colon \overline{X} \to X$ *is the projection map to the zeroth coordinate space and $\mathcal{V}$ is the vertex set of X.*

(3) $(\overline{X}, \overline{f})$ *and* (Σ_X, σ_X) *have the same entropy.*

Lemma 2.14 ([15, 1.6]). *If (X, f) satisfies all six Axioms, and I is an interval in X, then $X \subset f^m(I)$ for some $m \geq 0$.*

We remark that the proof of [15, 1.6] still works in our topological setting. Then for all $I_{i,j}, I_{k,l} \in \mathcal{A}$, $I_{k,l} \subset f^m(I_{i,j})$ for some positive integer m, and we have the following proposition.

Proposition 2.15. *If (X, f) satisfies all six Axioms, then (Σ_X, σ_X) is a mixing SFT.*

Examples 2.16. Let (X, f) and (Y, h) be as in Example 2.7. Recall that $\tilde{f}\colon \mathcal{E}_X \to \mathcal{E}_X^*$ and $\tilde{h}\colon \mathcal{E}_Y \to \mathcal{E}_Y^*$ are given by

$$\tilde{f}\colon e_1 \mapsto e_1 e_2, \ e_2 \mapsto e_1 e_2,$$

$$\tilde{h}\colon a \mapsto cabd, \ b \mapsto dc, \ \text{and } c \mapsto abc, \ d \mapsto ab.$$

The SFT covers of $(\overline{X}, \overline{f})$ and $(\overline{Y}, \overline{h})$ are given by the following matrices

$$M_X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } M_Y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Example 2.17. We need the Flattening Axiom in Lemma 2.14 and Proposition 2.15. Let Z be a wedge product of two circles a and b, and $g\colon Z \to Z$ given by $a \mapsto aa$ and $b \mapsto bb$. Then (Z, g) does not satisfy the Flattening Axiom, and the adjacency matrix M_Z for $(\mathcal{E}, \tilde{g})$ is $M_Z = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. So (Σ_Z, σ_Z) is not irreducible.

3. Shift equivalence

We define shift equivalence of directed graphs with graph maps, and show that conjugacy of branched solenoids is equivalent to shift equivalence of their presentations and that certain conjugacies of solenoids lift uniquely to conjugacies of associated SFT covers.

Definition 3.1. Suppose that X and Y are directed graphs, and that $f\colon X \to X$ and $g\colon Y \to Y$ are graph maps. We say that f and g are **shift equivalent with lag** m if there are continuous maps $r\colon X \to Y$, $s\colon Y \to X$ and a positive integer m such that

$$r \circ f = g \circ r, \quad f \circ s = s \circ g, \quad f^m = s \circ r, \quad g^m = r \circ s.$$

If (X, f) and (Y, g) are shift equivalent with lag m by continuous maps $r\colon X \to Y$ and $s\colon Y \to X$, then define induced maps $\overline{r}\colon \overline{X} \to \overline{Y}$ and $\overline{s}\colon \overline{Y} \to \overline{X}$ by

$$\overline{r}\colon (x_0, \ldots, x_m, x_{m+1}, \ldots) \mapsto (r(x_m), r(x_{m+1}), \ldots)$$
$$\overline{s}\colon (y_0, y_1, y_2, \ldots) \mapsto (s(y_0), s(y_1), s(y_2), \ldots).$$

We can easily check that $\overline{r}$ and $\overline{s}$ are topological conjugacies of $\overline{f}$ and $\overline{g}$ such that $\overline{s} \circ \overline{r} = Id$ on $\overline{X}$ and $\overline{r} \circ \overline{s} = Id$ on $\overline{Y}$.

Remark 3.2. It is possible that the shift equivalence map $r\colon X \to Y$ is not a graph map, that is, a vertex of X may not be mapped to a vertex of Y.

Lemma 3.3. *Suppose that (X, f) and (Y, g) satisfy the Markov Axiom, and they are shift equivalent to each other with lag m by continuous maps $r\colon X \to Y$ and $s\colon Y \to X$. Then the vertex sets $\mathcal{V}_X$ and $\mathcal{V}_Y$ of X and Y, respectively, can be enlarged to $\mathcal{V}_X'$ and $\mathcal{V}_Y'$, respectively, so that*

$$f(\mathcal{V}_X') \subset \mathcal{V}_X', \ g(\mathcal{V}_Y') \subset \mathcal{V}_Y', \ r(\mathcal{V}_X') \subset \mathcal{V}_Y', \ and \ s(\mathcal{V}_Y') \subset \mathcal{V}_X'.$$

Proof. For the vertex sets $\mathcal{V}_X$ and $\mathcal{V}_Y$, let $\mathcal{V}_Y' = \mathcal{V}_Y \cup r(\mathcal{V}_X)$ be the set of enlarged vertices in Y and $\mathcal{V}_X' = \mathcal{V}_X \cup s(\mathcal{V}_Y')$. Then

$$g(\mathcal{V}_Y') \subset g(\mathcal{V}_Y) \cup g \circ r(\mathcal{V}_X) \subset \mathcal{V}_Y \cup r \circ f(\mathcal{V}_X) \subset \mathcal{V}_Y \cup r(\mathcal{V}_X) = \mathcal{V}_Y',$$
$$f(\mathcal{V}_X') \subset f(\mathcal{V}_X) \cup f \circ s(\mathcal{V}_Y') \subset \mathcal{V}_X \cup s \circ g(\mathcal{V}_Y') \subset \mathcal{V}_X \cup s(\mathcal{V}_Y') = \mathcal{V}_X',$$
$$r(\mathcal{V}_X') = r(\mathcal{V}_X) \cup r \circ s(\mathcal{V}_Y') = r(\mathcal{V}_X) \cup g^m(\mathcal{V}_Y') \subseteq \mathcal{V}_Y', \ and$$
$$s(\mathcal{V}_Y') = s(\mathcal{V}_Y) \cup f^m(\mathcal{V}_X) \subseteq \mathcal{V}_X'$$

prove the Lemma. $\qquad \square$

Lemma 3.4 (Ladder Lemma). *Suppose that (X, f) and (Y, g) satisfy all Axioms except possibly the Flattening Axiom, and that $\phi\colon \overline{X} \to \overline{Y}$ is a continuous map such that $\overline{g} \circ \phi = \phi \circ \overline{f}$. Then there is a continuous map $r\colon X \to Y$ and a nonnegative integer n such that $g \circ r = r \circ f$ and $\phi(x_0, x_1, \ldots) = (r(x_n), r(x_{n+1}), \ldots)$.*

Remark 3.5. Williams ([15, §3]) proved the Ladder Lemma under the hypotheses that f and g are nonwandering expansions which are immersions of differentiable branched 1-manifolds and satisfy the Flattening Axiom. Our assumptions are weaker than his conditions as we don't need the Flattening Axiom, but the ideas of the proof are essentially those given by Williams. The additional complication of our setting is addressed in Lemma 2.9.

Let X_i and Y_i be the ith coordinate spaces of $\overline{X}$ and $\overline{Y}$, respectively, and π_i the projection map from the branched solenoids onto their ith coordinate spaces.

Lemma 3.6. *There is a positive integer n such that, for $a, b \in \overline{X}$, if $\pi_n(a) = \pi_n(b)$, then $\pi_0 \circ \phi(a) = \pi_0 \circ \phi(b)$.*

Proof. By Lemma 2.9, we can choose $\epsilon > 0$ and $l \in \mathbb{N}$ such that, for all $x, y \in Y$, if $g^l(x) \neq g^l(y)$, then there exists a nonnegative integer K such that

$$d(g^K(x), g^K(y)) \geq \epsilon.$$

Choose a nonnegative integer k and $\delta > 0$ such that, for $a = (a_0, a_1, \dots)$ and $b = (b_0, b_1, \dots) \in \overline{X}$, $a_k = b_k$ implies $d(a, b) < \delta$, and $d(a, b) < \delta$ implies $d(\phi(a), \phi(b)) < \epsilon$.

Now suppose $a_k = b_k$. From $d(\overline{f}^m(a), \overline{f}^m(b)) \leq d(a, b) < \delta$ for every nonnegative integer m, we have

$$d(\phi \circ \overline{f}^m(a), \phi \circ \overline{f}^m(b)) = d(\overline{g}^m \circ \phi(a), \overline{g}^m \circ \phi(b)) < \epsilon,$$

and if $x_0 = \pi_0 \circ \phi(a)$ and $y_0 = \pi_0 \circ \phi(b)$, then for every nonnegative integer m

$$d(g^m(x_0), g^m(y_0)) < \epsilon.$$

Therefore we have that $a_k = b_k$ implies $g^l \circ \pi_0 \circ \phi(a) = g^l \circ \pi_0 \circ \phi(b)$, and equivalently $a_{k+l} = b_{k+l}$ implies $\pi_0 \circ \phi(a) = \pi_0 \circ \phi(b)$.   □

Proof of Ladder Lemma. Let n be as in Lemma 3.6. Define $r_k \colon X \to Y$ by $r_k = \pi_k \circ \phi \circ \pi_{n+k}^{-1}$. Then r_k is well-defined by our choice of n. Now show that $r_k = r_0$ for every positive integer k. For $x = (x_0, \dots, x_n, \dots) \in \overline{X}$,

$$\begin{aligned}
r_k(x_n) &= \pi_k \circ \phi \circ \pi_{n+k}^{-1}(x_n) = \pi_k \circ \phi \circ \overline{f}^k(x) \\
&= \pi_k \circ \overline{g}^k \circ \phi(x) = \pi_0 \circ \phi(x) = \pi_0 \circ \phi \circ \pi_n^{-1}(x_n) \\
&= r_0(x_n).
\end{aligned}$$

To show the continuity of $r = r_0$, let $\delta > 0$ and $\epsilon > 0$ be as in Lemma 3.6, and choose $\delta' > 0$ such that if a_n, b_n are elements in X with $d(a_n, b_n) < \delta'$, then there exist $a, b \in \overline{X}$ with $\pi_n(a) = a_n$ and $\pi_n(b) = b_n$ such that $d(a, b) < \delta$. Then we have

$$d(a_n, b_n) < \delta' \implies d(\pi_0 \circ \phi(a), \pi_0 \circ \phi(b)) = d(r(a_n), r(b_n)) < \epsilon,$$

and $r \colon X \to Y$ is continuous. That $\phi(x_0, \dots, x_n, x_{n+1}, \dots) = (r(x_n), r(x_{n+1}), \dots)$ is trivial by the construction of $r \colon X \to Y$.   □

The proof of the following proposition is easy from the Ladder Lemma. So we omit the proof.

Proposition 3.7 ([15, Theorem 3.3]). *Suppose that (X, f) and (Y, g) satisfy all Axioms except possibly the Flattening Axiom. Then $\phi \colon \overline{X} \to \overline{Y}$ is a topological conjugacy if and only if there exists a shift equivalence (r, s) of f and g such that $\phi = \overline{r}$.*

Let (Σ_X, σ_X) and (Σ_Y, σ_Y) be the SFT covers of $(\overline{X}, \overline{f})$ and $(\overline{Y}, \overline{g})$ defined by $(\mathcal{E}_X, \tilde{f})$ and $(\mathcal{E}_Y, \tilde{g})$, respectively, as in §2.

Theorem 3.8. *Suppose that (X, f) and (Y, g) satisfy Axioms $3, 4, 5$, and they are shift equivalent to each other with lag m by graph maps r and s. Then the conjugacy $\overline{r} \colon \overline{X} \to \overline{Y}$ lifts to a unique conjugacy $\tilde{r}$ of (Σ_X, σ_X) and (Σ_Y, σ_Y).*

Proof. We will define a sliding block code $\phi_r \colon \Sigma_X \to \Sigma_Y$ induced by r, and show that ϕ_r is the required conjugacy $\tilde{r}$.

Let $\mathcal{E}_X$ and $\mathcal{E}_Y$ denote the edge sets of X and Y, respectively, $\mathcal{A}_X = \{I_{i,j}\}$ and $\mathcal{A}_Y = \{J_{k,l}\}$ the alphabets of (Σ_X, σ_X) and (Σ_Y, σ_Y), respectively, and $p_X \colon \Sigma_X \to \overline{X}$ and $p_Y \colon \Sigma_Y \to \overline{Y}$ the semiconjugacies. Then $I_{i,j}$ is a subset of $e_i \in \mathcal{E}_X$ such that

$f(I_{i,j}) = e_{i,j} \in \mathcal{E}_X$ (ignoring the direction). Note that if $a = \cdots I_{a(-1)}I_{a(0)}I_{a(1)} \cdots \in \Sigma_X$, $x_a = (x_0, x_1, \dots) = p_X(a) \in \overline{X}$, and $I_{a(i)} \subset e_{a_i}$, then $x_i \in I_{a(-i)}$ and $f^i(x_0) \in I_{a(i)}$ for every nonnegative integer i.

Let $\mathcal{C}_X = \{a \in \Sigma_X \mid p_X^{-1} \circ p_X(a) = \{a\}\}$ and $\mathcal{C}_Y = \{b \in \Sigma_Y \mid p_Y^{-1} \circ p_Y(b) = \{b\}\}$. Then by Proposition 2.13, $\mathcal{C}_X$ and $\mathcal{C}_Y$ are dense in Σ_X and Σ_Y, respectively.

Step 1. Show that $\overline{r} \circ p_X(\mathcal{C}_X) = p_Y(\mathcal{C}_Y)$.

By Proposition 2.13, $a \in \mathcal{C}_X$ if and only if $\pi_0 \circ p_X(a) \notin \bigcup\limits_{n=0}^{\infty} f^{-n-1}(\mathcal{V}_X)$ where π_i is the projection map from the branched solenoids to their ith coordinate spaces. So we have $\overline{f}^m \circ p_X(\mathcal{C}_X) = p_X(\mathcal{C}_X)$.

If $\overline{r} \circ p_X(a) \notin p_Y(\mathcal{C}_Y)$ for some $a \in \mathcal{C}_X$, then $r \circ \pi_m \circ p_X(a) \in \bigcup\limits_{n=0}^{\infty} g^{-n-1}(\mathcal{V}_Y)$ and

$$g^{n+1} \circ r \circ \pi_m \circ p_X(a) = r \circ f^{n+1} \circ \pi_m \circ p_X(a) \in \mathcal{V}_Y$$

for some $n \geq 0$. Since the shift equivalence maps r and s are graph maps by Lemma 3.3 and $s \circ r = f^m$, we have

$$s \circ r \circ f^{n+1} \circ \pi_m \circ p_X(a) = f^{n+m+1} \circ \pi_m \circ p_X(a) = f^{n+1} \circ \pi_0 \circ p_X(a) \in \mathcal{V}_X,$$

a contradiction. Hence we have $\overline{r} \circ p_X(\mathcal{C}_X) \subset p_Y(\mathcal{C}_Y)$. By the same argument, we can show that $\overline{s} \circ p_Y(\mathcal{C}_Y) \subset p_X(\mathcal{C}_X)$. Then $\overline{r} \circ \overline{s} = \overline{g}^m$ and $\overline{g}^m(\mathcal{C}_Y) = \mathcal{C}_Y$ imply that

$$p_Y(\mathcal{C}_Y) = \overline{r} \circ \overline{s} \circ p_Y(\mathcal{C}_Y) \subset \overline{r} \circ p_X(\mathcal{C}_X)$$

Therefore we have $\overline{r} \circ p_X(\mathcal{C}_X) = p_Y(\mathcal{C}_Y)$.

Now we have a well-defined bijective map $(p_Y|_{\mathcal{C}_Y})^{-1} \circ \overline{r} \circ p_X|_{\mathcal{C}_X} \colon \mathcal{C}_X \to \mathcal{C}_Y$. This map will define $\tilde{r}$ on $\mathcal{C}_X$.

Step 2. Find a block map $\Phi_r \colon \mathcal{B}_{m+1}(\mathcal{C}_X) \to \mathcal{A}_Y$ where $\mathcal{B}_{m+1}(\mathcal{C}_X)$ is the set of all $(m+1)$-blocks in $\mathcal{C}_X$ such that for every $a = \cdots I_{a(-m+i)} \cdots I_{a(-1+i)}I_{a(i)} \cdots \in \mathcal{C}_X$ $(p_Y|_{\mathcal{C}_Y})^{-1} \circ \overline{r} \circ p_X|_{\mathcal{C}_X}(a)_i = \Phi_r(I_{a(-m+i)} \cdots I_{a(i)})$.

For $a = \cdots I_{a(-m)} \cdots I_{a(-1)}I_{a(0)} \cdots \in \mathcal{C}_X$, let $x_a = (x_0, x_1, \dots) = p_X(a) \in \overline{X}$ and $y_a = (y_0, y_1, \dots) = \overline{r}(x_a) \in p_Y(\mathcal{C}_Y)$. Then $x_i \in I_{a(-i)} \subset e_{a(-i)}$, $y_i = r(x_{i+m})$, and there exists a unique $\alpha = \cdots J_{\alpha(-1)}J_{\alpha(0)} \cdots \in \mathcal{C}_Y$ such that

$$(p_Y|_{\mathcal{C}_Y})^{-1} \circ \overline{r} \circ p_X|_{\mathcal{C}_X}(a) = \alpha \text{ and } p_Y(\alpha) = y_a.$$

Let $\{I_{e,j}^{(k)}\}$ be the partition of $e = e_{a(-m)} \in \mathcal{E}_X$ for f^k, $1 \leq k \leq m+1$. Then each $I_{e,j}^{(k)}$ is contained in a unique $I_{e,j'}^{(k-1)}$, and we have a unique descending sequence

$$e \supset I_{a(-m)} = I_{e,u(1)}^{(1)} \supset I_{e,u(2)}^{(2)} \supset \cdots \supset I_{e,u(m+1)}^{(m+1)}$$

such that $f^k(I_{e,u(k)}^{(k)}) = e_{a(-m+k)} \in \mathcal{E}_X$, $f^k(I_{e,u(k+1)}^{(k+1)}) = I_{a(-m+k)} \subset e_{a(-m+k)}$, and $x_m \in I_{e,u(m+1)}^{(m+1)}$.

Since the shift equivalence maps r and s are graph maps and $s \circ r = f^m$, $f^m(I_{e,u(m)}^{(m)}) = e_{a(0)}$ and $f^{m+1}(I_{e,u(m+1)}^{(m+1)}) = e_{a(1)}$ imply that $r(I_{e,u(m)}^{(m)})$ is contained in a unique edge ϵ in Y and that $r(I_{e,u(m+1)}^{(m+1)})$ is contained in a unique path J such that $J \subset \epsilon$ and $g(J) \in \mathcal{E}_Y$. Define a block map $\Phi_r \colon \mathcal{B}_{m+1}(\mathcal{C}_X) \to \mathcal{A}_Y$ by

$$I_{a(-m)} \cdots I_{a(-1)}I_{a(0)} \mapsto J.$$

Then the sliding block code ϕ_r induced by Φ_r maps a to $\beta = \cdots J_{\beta(0)}J_{\beta(1)} \cdots$ with $J_{\beta(i)} = \Phi_r(I_{a(-m+i)} \cdots I_{a(i)})$.

To prove that $\phi_r = (p_Y|_{\mathcal{C}_Y})^{-1} \circ \overline{r} \circ p_X|_{\mathcal{C}_X}$, we need only show that $p_Y(\beta) = y_a$, that is, $y_i = r(x_{i+m}) \in J_{\beta(-i)}$. From the descending sequence for x_{i+m}

$$(2) \qquad e_{a(-i-m)} \supset I_{a(-i-m)} \supset I^{(2)}_{a(-i-m),u(2)} \supset \cdots \supset I^{(m+1)}_{a(-i-m),u(m+1)} \ni x_{i+m},$$

$J_{\beta(-i)}$ is the unique path in Y such that $r(I^{(m+1)}_{a(-i-m),u(m+1)}) \subset J_{\beta(-i)}$ and $g(J_{\beta(-i)}) \in \mathcal{E}_Y$. So $y_i = r(x_{i+m})$ is contained in $J_{\beta(-i)}$, and $p_Y(\beta) = y_a$. Therefore $\phi_r(a) = \beta = \alpha = (p_Y|_{\mathcal{C}_Y})^{-1} \circ \overline{r} \circ p_X|_{\mathcal{C}_X}(a)$ by the definition of $\mathcal{C}_Y$, and this proves $\phi_r|_{\mathcal{C}_X} = (p_Y|_{\mathcal{C}_Y})^{-1} \circ \overline{r} \circ p_X|_{\mathcal{C}_X}$.

Step 3. Define a block map $\Psi_s \colon \mathcal{B}_{m+1}(\mathcal{C}_Y) \to \mathcal{A}_X$ with ψ_s defined on $\mathcal{C}_Y$ by $\psi_s(\alpha)_i = \Psi_s(J_{\alpha(i)} \cdots J_{\alpha(i+m)})$ such that $\psi_s \circ \phi_r = Id$ on $\mathcal{C}_X$.

We define Ψ_s from $s \colon Y \to X$ just as we defined Φ_r from r in Step 2. For $\alpha = \cdots J_{\alpha(-i)} J_{\alpha(-i+1)} \cdots \in \mathcal{C}_Y$ with $J_{\alpha(-i)} \subset \epsilon$, let $\{J^{(k)}_{\epsilon,l(k)}\}$ be the partition of $\epsilon \in \mathcal{E}_Y$ for g^k, $1 \le k \le m+1$, so that $g^k(J^{(k)}_{\epsilon,l(k)}) = \epsilon_{\alpha(-i+k)} \in \mathcal{E}_Y$ and $g^k(J^{(k+1)}_{\epsilon,l(k+1)}) = J_{\alpha(-i+k)} \subset \epsilon_{\alpha(-i+k)}$. Then we have a unique descending sequence

$$(3) \qquad\qquad \epsilon \supset J_{\alpha(-i)} = J^{(1)}_{\epsilon,v(1)} \supset \cdots \supset J^{(m+1)}_{\epsilon,v(m+1)}$$

such that, for $y = p_Y(\alpha)$, we have $y_i \in J^{(m+1)}_{\epsilon,v(m+1)}$ and $s(J^{(m)}_{\epsilon,v(m)})$ is contained in a unique path I in X such that $f(I) \in \mathcal{E}_X$. We define $\Psi_s \colon \mathcal{B}_{m+1}(\mathcal{C}_Y) \to \mathcal{A}_X$ by

$$J_{\alpha(-i)} \cdots J_{\alpha(-i+m)} \mapsto I.$$

Let $p_X|_{\mathcal{C}_X}(a) = x$, $\overline{r}(x) = y$, and $(p_Y|_{\mathcal{C}_Y})^{-1}(y) = \alpha$. If $x_{i+m} \in I^{(m+1)}_{a(-i-m),u(m+1)}$ and $y_i \in J^{(m+1)}_{\epsilon,v(m+1)}$ as in the equations (2) and (3), then we have $J^{(m+1)}_{\epsilon,v(m+1)} \subset r(I^{(m+1)}_{a(-i-m),j(m+1)})$ and

$$s(J^{(m+1)}_{\alpha(-i),v(m+1)}) \subset s \circ r(I^{(m+1)}_{a(-i-m),j(m+1)}) = I_{a(-i)} \in \mathcal{A}_X.$$

Therefore $\psi_s \circ \phi_r \colon \mathcal{C}_X \to \mathcal{C}_X$ is the sliding block code with memory m and anticipation m induced by a block map defined by

$$I_{a(-i-m)} \cdots I_{a(-i)} \cdots I_{a(-i+m)} \mapsto I_{a(-i)},$$

and $\psi_s \circ \phi_r = Id$ on $\mathcal{C}_X$.

Step 4. Deduce that ϕ_r gives the required conjugacy $\tilde{r}$.

Because ϕ_r maps $\mathcal{C}_X$ onto $\mathcal{C}_Y$, and these sets are dense in Σ_X and Σ_Y, it follows that ϕ_r maps Σ_X onto Σ_Y. Similarly ψ_s maps Σ_Y onto Σ_X. Since the continuous maps $Id|_{\Sigma_X}$ and $\psi_s \circ \phi_r$ agree on the dense set $\mathcal{C}_X$, we have $Id|_{\Sigma_X} = \psi_s \circ \phi_r$ on Σ_X, and so ϕ_r is a conjugacy.

That ϕ_r is a lift of $\overline{r}$ follows because $p_Y \circ \phi_r = \overline{r} \circ p_X$ on the dense set $\mathcal{C}_X$. The lifting is unique for it is uniquely determined on the dense set $\mathcal{C}_X$. $\qquad\square$

Remark 3.9. It is necessary to assume the shift equivalence by graph maps. See Examples 4.8 and 5.1.

4. Graph algorithm

Suppose that (X, f) is a presentation of a solenoid satisfying all six Axioms. Given a finite subset $\mathcal{O}$ of X such that $f(\mathcal{O}) = \mathcal{O}$, we will construct a new presentation $(X_\mathcal{O}, f_\mathcal{O})$ such that $(\overline{X}_\mathcal{O}, \overline{f}_\mathcal{O})$ is topologically conjugate to $(\overline{X}, \overline{f})$. For this purpose, we will give a graph algorithm which takes the given presentation (X, f) and $\mathcal{O}$ to produce a new presentation $(X_\mathcal{O}, f_\mathcal{O})$ and shift equivalence maps $\rho_\mathcal{O} \colon (X_\mathcal{O}, f_\mathcal{O}) \to (X, f)$ and $\psi_\mathcal{O} \colon (X, f) \to (X_\mathcal{O}, f_\mathcal{O})$.

Notation 4.1. By a *path* or *directed path l*, we mean an equivalence class of locally one-to-one continuous maps $\gamma \colon [0, 1] \to X$ where $\gamma_1 \sim \gamma_2$ if and only if there is an order-preserving homeomorphism $h_{1,2} \colon [0, 1] \to [0, 1]$ such that $\gamma_1(t) = \gamma_2 \circ h_{1,2}(t)$ for all $t \in [0, 1]$. So a path $l = [\gamma]$ has the initial point $\gamma(0)$ and the terminal point $\gamma(1)$. By abuse of notation, when we say that a path contains a set, we mean that the image of the path contains the set.

Let $\mathcal{P}$ be the set of directed paths $l \subset X$ such that the boundary points of l are contained in $\mathcal{O}$, and the interior of l does not contain any point of $\mathcal{O}$. Let l be a directed path in $\mathcal{P}$. Since $f(\mathcal{O}) = \mathcal{O}$, if we consider $f(l)$ as a continuous map $L \colon [0, 1] \to X$, then the interval $[0, 1]$ can be represented as a union of subintervals $[0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, 1]$ such that $L(a_i) \in \mathcal{O}$ and $L((a_{i-1}, a_i)) \cap \mathcal{O} = \emptyset$ for every $0 < i < n$. Hence for the collection of $[a_{i-1}, a_i]$ such that $L([a_{i-1}, a_i]) = l_i \in \mathcal{P}$, $f \colon X \to X$ induces a *wrapping rule* $\tilde{f}_\mathcal{P} \colon \mathcal{P} \to \mathcal{P}^*$ defined by $l \mapsto l_{1'} \cdots l_{n'}$. We write $l_i \in \tilde{f}(l)$ if $l_i \in \mathcal{P}$ is one of these factors of $\tilde{f}(l)$. (We remark that some factors of $\tilde{f}(l)$ may not be paths, because $\tilde{f}(l)$ need not be locally one-to-one.)

Our first task is to find a minimal set of directed paths $\mathcal{P}_\mathcal{O}$ such that $\mathcal{P}_\mathcal{O}$ is a finite subset of $\mathcal{P}$, $X \subset \bigcup_{l \in \mathcal{P}_\mathcal{O}} l$, $\tilde{f}(l) \in \mathcal{P}_\mathcal{O}^*$ for every $l \in \mathcal{P}_\mathcal{O}$, and there exists a positive integer k such that $l_1 \in \tilde{f}^k(l_2)$ for all $l_1, l_2 \in \mathcal{P}_\mathcal{O}$.

Let's denote $\mathcal{I}_m$ as the set of directed paths whose boundary points are contained in $f^{-m}(\mathcal{O})$, and whose interior does not have any point in $f^{-m}(\mathcal{O})$. Since we assumed all six Axioms, there is a positive integer n such that $f^n(e) \supseteq X$ for every edge e of X from Lemma 2.14. So each edge e contains at least one point of $f^{-n}(\mathcal{O})$, and $\mathcal{I}_m$ is a finite set for $m \geq n$.

We have the following lemma from the Flattening Axiom.

Lemma 4.2. *There exists a positive integer $N \geq n$ such that the interior of each edge contains at least one point of $f^{-N}(\mathcal{O})$, and for each vertex $v \in X$*

(1) *if $v \notin f^{-N}(\mathcal{O})$, then there exists a path $l_v \in \mathcal{P}$, unique up to direction, such that for every path $I \in \mathcal{I}_N$ which contains v as an interior point, either $f^N(I) \notin \mathcal{P}$ or $f^N(I) = l_v^{\pm 1} \in \mathcal{P}$ where ± 1 denotes the direction, and*

(2) *if $v \in f^{-N}(\mathcal{O})$, then there exist paths $l_i, l_t \in \mathcal{P}$, unique up to direction, such that if J_1 and J_2 are elements of $\mathcal{I}_N$ such that v is the terminal point of J_1 and the initial point of J_2, then $f^N(J_1 J_2) = (l_i l_t)^{\pm 1}$.*

Fix N satisfying the statement of Lemma 4.2. Let $\mathcal{I} = \{I \in \mathcal{I}_N \mid f^N(I) \in \mathcal{P}\}$. Then $I \in \mathcal{I}_N \backslash \mathcal{I}$ if and only if $f^N(I)$ is not locally one-to-one. Each $l \in \mathcal{P}$ can be represented as a product of $I_i \in \mathcal{I}_N$ such that the initial point of I_1 is the initial point of l, the terminal point of I_i is the initial point of I_{i+1} for $1 \leq i < j(l)$, and the terminal point of $I_{j(l)}$ is the terminal point of l so that some $I_i \in \mathcal{I}$, some $I_j \in \mathcal{I}_N \backslash \mathcal{I}$, and $\tilde{f}^N(l) = f^N(I_{i(1)}) \cdots f^N(I_{i(l)})$ where $I_{i(k)} \in \mathcal{I}$. Therefore each

factor of $\tilde{f}^{N+i}(l)$ which is a path in X is an element of $f^i(\mathcal{I})$ for every $l \in \mathcal{P}$ and $i \geq 0$.

Lemma 4.3. *Suppose $l = f^N(I) \in \mathcal{P}$ for some $I \in \mathcal{I}$. Then every factor of $\tilde{f}^i(l)$ is an element of $f^N(\mathcal{I})$ for every nonnegative integer i.*

Proof. Clearly every factor of $\tilde{f}^{(i)}(l)$ is an element of $f^N(\mathcal{I}_N)$. We must check that every factor is locally one to one. First consider the case that the image of I is a subset of an edge of X. Assume that I is represented as a product $J_1 \cdots J_{j(I)}$ such that each $J_j \in \mathcal{I}_{N+i}$, and $f(J_j) \notin \mathcal{I}$ for some j. Then $f(J_j) \notin \mathcal{I}$ implies that $f^{N+i}(J_j)$ is not locally one-to-one on the image of J_j, and so $f^{N+1}(I)$ is not locally one-to-one. This contradicts the Nonfolding Axiom as we chose the image of I to be a subset of an edge. So we have $f(J_j) \in \mathcal{I}$ for $1 \leq j \leq j(I)$.

Now suppose that I contains a vertex v of X as an interior point. Let $I' \in \mathcal{I}$ be a subset of an edge. Then there is a positive integer k and a factor $J \in \mathcal{I}_{N+k}$ of I' such that $X \subset f^k(I')$ and $f^k(J)$ contains v as an interior point. By Lemma 4.2, we have $f^N(I) = f^{N+k}(J)^{\pm 1} \in \mathcal{P}$, and $f^{N+i}|_I = f^{N+k+i}|_J$ is locally one-to-one as J is a subset of an edge. Therefore factors of $\tilde{f}^i(f^N(I))$ are elements of $f^N(\mathcal{I})$ for all $i \geq 0$. □

Definition 4.4 (Closed finite path set up to direction). The directed paths which are elements of $\mathcal{I}$ come in pairs, where one path in a pair is the other with reversed direction. Make a choice of one path from each pair and let $\mathcal{I}_{or}$ be the collection of chosen directed paths. Define

$$\mathcal{P}_{\mathcal{O}} = \{f^N(I) \mid I \in \mathcal{I}_{or}\}.$$

Then $\mathcal{P}_{\mathcal{O}}$ is a finite subset of $\mathcal{P}$ as $\mathcal{I}$ is a finite set.

Proposition 4.5. *The set $\mathcal{P}_{\mathcal{O}}$ is the unique, up to the choice of direction, minimal subset of $\mathcal{P}$ satisfying the following conditions: $\tilde{f}(l) \in \mathcal{P}_{\mathcal{O}}^*$ for every $l \in \mathcal{P}_{\mathcal{O}}$, and $X \subset \bigcup_{l \in \mathcal{P}_{\mathcal{O}}} l$. There exists a positive integer k such that $l_1 \in \tilde{f}^k(l)$ for all $l_1, l \in \mathcal{P}_{\mathcal{O}}$.*

Proof. By Lemma 4.3, $\tilde{f}(l)$ or $\tilde{f}(l)^{-1}$ is contained in $\mathcal{P}_{\mathcal{O}}^*$ for every $l \in \mathcal{P}_{\mathcal{O}}$. It remains to check minimality.

Suppose that $l = f^N(I)$ for some $I \in \mathcal{I}$. Then by Lemma 2.14, there exists a positive integer $j \geq N$ such that $X \subset f^j(I)$. If $l_1 = f^N(I_1)$ such that $I_1 \in \mathcal{I}$ and the interior of I_1 is contained in an edge e_1, then there exists a subpath $J_1 \subset I$ such that $f^j(J_1) = I_1^{\pm 1}$, and we have $f^N(I_1) = l_1^{\pm 1}$ is a factor of $\tilde{f}^j(f^N(I))$.

Next suppose that $l_2 = f^N(I_2)$ where I_2 is a path in $\mathcal{I}$ such that I_2 contains a vertex v as an interior point. Let $I_3 \in \mathcal{I}$ be contained in an edge e, and $l_3 = f^N(I_3)$. Then for some $m > 0$, v is the image under f^m of an interior point of e, and by Lemma 4.2, $l_v^{\pm 1}$ is a factor of $\tilde{f}^{N+m}(l_3)$. This proves the minimality of $\mathcal{P}_{\mathcal{O}}$, and the uniqueness claim is also clear.

It is clear that $X \subset \bigcup_{l \in \mathcal{P}_{\mathcal{O}}} l$, and for all $l_1, l_2 \in \mathcal{P}_{\mathcal{O}}$ there exists a positive integer $k = k(1,2)$ such that $l_1^{\pm 1}$ is a factor of $\tilde{f}^k(l_2)$. Then the number k can be chosen uniformly because for every $I \in \mathcal{I}$ such that the image of I is contained in an edge, if $l = f^N(I)$, then $l^{\pm 1}$ is a factor of $f^m(l')$ for every $l' \in \mathcal{P}_{\mathcal{O}}$ and large m. □

Definition 4.6 (Construction of new presentation). The new directed graph $X_{\mathcal{O}}$ defined by the set $\mathcal{O}$ has n vertices and m edges where n is the cardinality of $\mathcal{O}$ and

m is the cardinality of $\mathcal{P}_\mathcal{O}$. The set of vertices $\mathcal{V}_\mathcal{O}$ of $X_\mathcal{O}$ corresponds to $\mathcal{O}$, and the set of edges $\mathcal{E}_\mathcal{O}$ corresponds to $\mathcal{P}_\mathcal{O}$ by the following rule: If $l_i \in \mathcal{P}_\mathcal{O}$ is a directed path from $v_{i,1}$ to $v_{i,2}$ in X represented by a continuous map $\gamma_{l_i}\colon [0,1] \to X$ as in Notation 4.1, then $e_i \in \mathcal{E}_\mathcal{O}$ is a directed edge from $v_{i,1}$ to $v_{i,2}$ represented by a continuous map $\gamma_i\colon [0,1] \to e_i$ such that $\gamma_i|_{(0,1)}$ is a homeomorphism, and l_i^{-1} corresponds to e_i^{-1} defined by $\gamma_i^{-1}\colon [0,1] \to e_i$ such that $\gamma_i^{-1}(t) = \gamma_i(1-t)$. So there is a natural projection $\rho_\mathcal{O}\colon X_\mathcal{O} \to X$ defined by $\gamma_{l_i} \circ (\gamma_i)^{-1}\colon e_i \to [0,1] \to l_i$, and the graph map $f_\mathcal{O}\colon X_\mathcal{O} \to X_\mathcal{O}$ is induced by $f\colon X \to X$ satisfying $\rho_\mathcal{O} \circ f_\mathcal{O} = f \circ \rho_\mathcal{O}$. Hence if $\tilde{f}_\mathcal{P}\colon l_i \mapsto l_{i,1}^{s(1)} \cdots l_{i,m(i)}^{s(m(i))} \in \mathcal{P}_\mathcal{O}^*$ for $l_i \in \mathcal{P}_\mathcal{O}$, then $\tilde{f}_\mathcal{O}\colon e_i \mapsto e_{i,1}^{s(1)} \cdots e_{i,m(i)}^{s(m(i))} \in \mathcal{E}_\mathcal{O}^*$ where $\rho_\mathcal{O}(e_i) = l_i \in \mathcal{E}_\mathcal{O}$ and $s(i)$ denotes the direction.

Remark 4.7. Suppose that $\mathcal{P}_\mathcal{O}'$ is $\mathcal{P}_\mathcal{O}$ with a different choice of directions of paths. Let $(X_\mathcal{O}, f_\mathcal{O})$ be defined by $\mathcal{P}_\mathcal{O}$, and $(X_\mathcal{O}', f_\mathcal{O}')$ defined by $\mathcal{P}_\mathcal{O}'$. Then the two graphs are the same except that the directions of some corresponding edges might be reversed, and it is easy to see that $f_\mathcal{O}$ and $f_\mathcal{O}'$ are shift equivalent by graph maps.

Example 4.8. Let X be a wedge of two circles a, b with a unique vertex p, and $f\colon X \to X$ defined by $a \mapsto aab$ and $b \mapsto ab$. So (X, f) is given by the following diagram, in which p is the vertex of X and $\{q, r\}$ is a period 2 orbit.
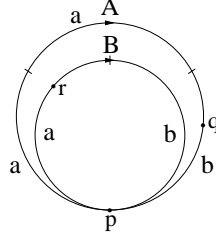

The set of directed paths $\mathcal{P}$ defined by the indicated periodic orbit $\{q, r\}$ is $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \alpha^{-1}, \beta^{-1}, \gamma^{-1}, \delta^{-1}, \epsilon^{-1}, \zeta^{-1}\}$ where α is the circle A based at q, β is the path from q through p to r, γ is from r through p to q, δ is the circle B based at r, ϵ is the path from q through p to r with reverse direction to B, and ζ is the path from r through p with reverse direction to B to q.

Every edge of X has at least two points of $f^{-1}(\{q, r\})$, and it is not difficult to check that $f(\mathcal{I}) = \{\alpha, \beta, \gamma, \alpha^{-1}, \beta^{-1}, \gamma^{-1}\}$. So $\mathcal{P}_{\{q,r\}}$ is $\{\alpha, \beta, \gamma\}$ up to the choice of direction, and the induced wrapping rule $\tilde{f}_\mathcal{P}\colon \mathcal{P} \to \mathcal{P}^*$ is given by

$$\alpha \mapsto \gamma\alpha\beta, \quad \beta \mapsto \gamma, \quad \gamma \mapsto \beta\gamma\alpha\beta.$$

Hence the new graph $X_{\{q,r\}}$ defined by $\{q, r\}$ is the following graph.
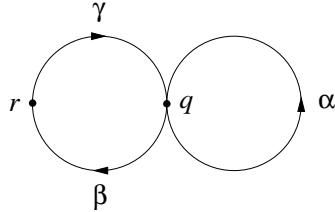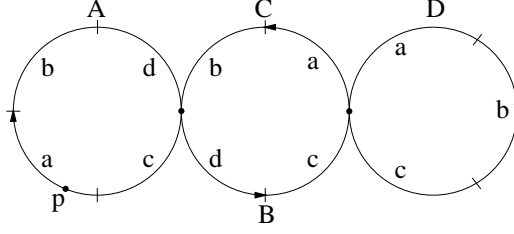
FIGURE 3. (Y, g) with a fixed point p

Remark 4.9. To compute the graph $X_{\mathcal{O}}$ and the map $f_{\mathcal{O}}$, we don't need to find the integer N and $\mathcal{I}$ given in Lemma 4.2. If we choose a path $l \in \mathcal{P}$, and iteratively apply $\tilde{f}$ to the factors of $\tilde{f}^n(l)$ which are paths, we will eventually generate a set of paths invariant under $\tilde{f}$, and this set will contain $\mathcal{P}_{\mathcal{O}}$.

Example 4.10. Suppose that (Y, g) is given by Figure 3, and p is a fixed point of g. Then the set $\mathcal{P}$ of directed paths based at p is an infinite set for if we call a_1 the path from p to the branch point in the edge a and a_2 the path from the branch point to p, then the paths $a_1 b \underbrace{d \cdots d}_{n} ca_2$ are in $\mathcal{P}$.

If we choose ℓ as the loop $a_1 a_2$ based at p, then $\tilde{g}(\ell) = a_1 bdca_2$ and $\tilde{g}^2(\ell) = a_1 bddca_2\, a_1 bca_2\, a_1 bca_2\, a_1 bca_2$.

For $e = a_1 bddca_2$ and $f = a_1 bca_2$, $\tilde{g} \colon e \mapsto efff$ and $f \mapsto ef$. So $\{e, f\}$ is a closed minimal subset of $\mathcal{P}$, and $\mathcal{P}_{\{p\}} = \{e, f\}$ by uniqueness. The new graph $(Y_{\{p\}}, g_{\{p\}})$ defined by $\{p\}$ is the following graph.
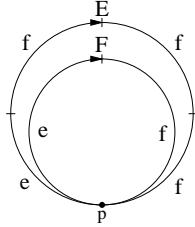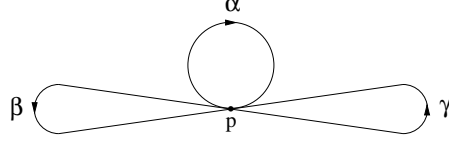

Example 4.11. Suppose that Z is given in the following graph, that $h \colon Z \to Z$ is given by $a \mapsto bab^{-1}$ and $b \mapsto aba^{-1}$, and that p is a fixed point.

Let a_1 denote the path from p to the branch point, a_2 the path from the branch point to p, and $\ell = a_2 ba_1$. Then $\tilde{h}(\ell) = a_2 b^{-1} a_1\, a_2 ba_2^{-1}\, a_1^{-1} ba_1$ and for $\alpha = a_2 b^{-1} a_1$, $\beta = a_2 ba_2^{-1}$ and $\gamma = a_1^{-1} ba_1$,

$$\tilde{h} \colon \alpha \mapsto \alpha\beta^{-1}\gamma, \quad \beta \mapsto \alpha\beta\alpha^{-1}, \quad \gamma \mapsto \gamma^{-1}\beta\gamma.$$

The new graph $Z_{\{p\}}$ defined by $\{p\}$ is a wedge of three circles as in the graph below.

Theorem 4.12. *Suppose that (X, f) satisfies all six Axioms, that $\mathcal{O}$ is a finite subset of X such that $f(\mathcal{O}) = \mathcal{O}$, and that $\mathcal{P}_{\mathcal{O}}$ and $(X_{\mathcal{O}}, f_{\mathcal{O}})$ are the minimal closed subset of $\mathcal{P}$ and the new presentation defined by $\mathcal{O}$, respectively. Then the natural projection $\rho_{\mathcal{O}} \colon X_{\mathcal{O}} \to X$ gives a conjugacy from $(\overline{X}_{\mathcal{O}}, \overline{f}_{\mathcal{O}})$ to $(\overline{X}, \overline{f})$.*

Proof. We will show that $\rho = \rho_{\mathcal{O}}$ is part of a shift equivalence, that is, we will define a continuous map $\psi \colon X \to X_{\mathcal{O}}$ and a positive integer m such that

$$f_{\mathcal{O}} \circ \psi = \psi \circ f, \ \rho \circ \psi = f^m, \text{ and } \psi \circ \rho = f_{\mathcal{O}}^m.$$

Recall $\mathcal{P}_{\mathcal{O}} = \{f^N(I) \mid I \in \mathcal{I}_{or}\}$ for some fixed choice of a subset $\mathcal{I}_{or}$ of $\mathcal{I}_N$ as in Definition 4.4. By the choice of N (from Lemma 4.2), every $I \in \mathcal{I}_{or}$ contains at most one vertex of X. It is trivial that if $x \in X \backslash f^{-N}(\mathcal{O})$ is contained in some $I \in \mathcal{I}_{or}$ and the interior of I does not contain any vertex of X, then I is the unique element in $\mathcal{I}_{or}$ which contains x.

If v is a vertex of X, then by Lemma 4.2, $v \in f^{-N}(\mathcal{O})$ or there is a neighborhood U_v of v such that U_v is the union of all paths in $\mathcal{I}$ containing v as an interior point and $f^N(U_v)$ is the image of $l_v \in \mathcal{P}_{\mathcal{O}}$. Hence, for every $x \in X \backslash f^{-N}(\mathcal{O})$, there is a unique path $l_x \in \mathcal{P}_{\mathcal{O}}$ such that if $x \in I \in \mathcal{I}_{or}$, then $f^N(I) = l_x^{\pm 1}$.

For $x \in X \backslash f^{-N}(\mathcal{O})$, let e_x be the edge of $X_{\mathcal{O}}$ corresponding to l_x. We will define $\psi(x)$ to be the appropriate point $x_{\mathcal{O}}$ in e_x satisfying $\rho \circ \psi(x) = f^N(x)$. Let $\gamma \colon [0, 1] \to e_x$ be the continuous function (in the equivalence class l_x) associated to e_x in the definition of $X_{\mathcal{O}}$ and ρ. Fix I in $\mathcal{I}_{or}$ such that $x \in I$. For the moment let I also denote a specific map $[0, 1] \to X$. Then there is a homeomorphism $h \colon [0, 1] \to [0, 1]$ such that $\rho \circ \gamma = f^N \circ I \circ h$. Let t be the unique number in $(0, 1)$ such that $I \circ h(t) = x$, and define $x_{\mathcal{O}} = \gamma(t)$. Then $\rho \circ \psi(x) = \rho(x_{\mathcal{O}}) = \rho \circ \gamma(t) = f^N \circ I \circ h(t) = f^N(x)$ as required.

For $x \in f^{-N}(\mathcal{O})$, we define $\psi(x)$ as the unique point in $X_{\mathcal{O}}$ which ρ maps to $f^N(x) \in \mathcal{O}$. Then ψ is continuous, $\rho \circ \psi = f^N$, and clearly $f \circ \rho = \rho \circ f_{\mathcal{O}}$ by Definition 4.6. Consequently $\rho \circ \psi \circ \rho = f^N \circ \rho = \rho \circ f_{\mathcal{O}}^N$. Therefore the two maps $\psi \circ \rho$ and $f_{\mathcal{O}}^N$ send any given edge (considered as a path) in $X_{\mathcal{O}}$ to paths which ρ sends to the same concatenation of elements of $\mathcal{I}$. Such a concatenation has a unique lifting under ρ, therefore $\psi \circ \rho = f_{\mathcal{O}}^N$.

It remains to show that $f_{\mathcal{O}} \circ \psi = \psi \circ f$. Because ρ is surjective, it suffices to show $f_{\mathcal{O}} \circ \psi \circ \rho = \psi \circ f \circ \rho$, and this is true because

$$f_{\mathcal{O}} \circ \psi \circ \rho = f_{\mathcal{O}}^{N+1} = (\psi \circ \rho) \circ f_{\mathcal{O}} = \psi \circ (\rho \circ f_{\mathcal{O}}) = \psi \circ (f \circ \rho).$$

Therefore $(X_{\mathcal{O}}, f_{\mathcal{O}})$ is shift equivalent to (X, f), and $\overline{\rho}_{\mathcal{O}}$ is a topologically conjugacy by Proposition 3.7. $\qquad\square$

Remarks 4.13.   (1) Theorem 4.12 requires the Flattening Axiom. See Example 4.17.
   (2) If $\mathcal{O}$ is not a subset of $\mathcal{V}_X$, then the shift equivalence maps ρ and ψ in Theorem 4.12 cannot be graph maps.

Remark 4.14 (Preperiodic sets). If $\mathcal{F}$ is a finite subset of X such that $f(\mathcal{F}) \subseteq \mathcal{F}$, then it is not difficult to apply the graph algorithm to $\mathcal{F}$ so that we have a finite graph with a graph map $(X_\mathcal{F}, f_\mathcal{F})$. And there is a positive integer $k = k_\mathcal{F}$ such that $f^k(\mathcal{F}) = f^{k+i}(\mathcal{F})$ for all $i \geq 0$. Let $\mathcal{O}_\mathcal{F} = f^k(\mathcal{F})$.

We get f and $f_\mathcal{F}$ shift equivalent just as before. Now it is not hard to check that $f_\mathcal{F}$ and $f_{\mathcal{O}_\mathcal{F}}$ are shift equivalent by graph maps. Hence the associated SFT covers of the next section will be conjugate for $\mathcal{F}$ and $\mathcal{O}_\mathcal{F}$, and that is why we only concern ourselves with sets $\mathcal{O}$ which are unions of periodic orbits.

Elementary presentation. One interesting application of the graph algorithm is the elementary presentations of solenoids. In [15, §5], Williams introduced an *elementary presentation* of a solenoid in which X is a wedge of circles and f leaves the unique branch point of X fixed. And he showed in [15, Theorem 5.2] that, for every generalized 1-dimensional solenoid $(\overline{X}, \overline{f})$, there exists an integer m such that $(\overline{X}, \overline{f^m})$ has an elementary presentation. We can improve his theorem by getting sharp bounds on m.

Suppose that (X, f) satisfies all six axioms, and that a is a fixed point of f^m. For (X, f^m), let $(X_{\{a\}}, f^m_{\{a\}})$ be the new presentation defined by $\{a\}$. Then the new graph $X_{\{a\}}$ has only one vertex a which is a fixed point by $f^m_{\{a\}} : X_{\{a\}} \to X_{\{a\}}$, and each edge in $X_{\{a\}}$ is homeomorphic to a circle. So $(X_{\{a\}}, f^m_{\{a\}})$ is an elementary presentation and $(\overline{X}, \overline{f^m})$ is conjugate to $(\overline{X}_{\{a\}}, \overline{f^m}_{\{a\}})$ by Theorem 4.12. More generally we have the following proposition:

Proposition 4.15. *For a given 1-solenoid, the minimal number of vertices in a presentation (X, f) is the minimal period of points in X. In particular, $(\overline{X}, \overline{f})$ has an elementary presentation if and only if $f : X \to X$ has a fixed point.*

Remark 4.16. Williams showed that two elementary presented solenoids $(\overline{Y}_1, \overline{g}_1)$ and $(\overline{Y}_2, \overline{g}_2)$ are topologically conjugate to each other if and only if the shift equivalence classes of $g_{1*} : \pi_1(Y_1, y_1) \to \pi_1(Y_1, y_1)$ and $g_{2*} : \pi_1(Y_2, y_2) \to \pi_1(Y_2, y_2)$ are the same where y_i is the unique branch point of Y_i for $i = 1, 2$ ([15, 7.3]). Proposition 4.15 extends the range of Williams' classification theorem.

Example 4.17. Theorem 4.12 and Proposition 4.15 require the Flattening Axiom.
Let X be a wedge of two circles a and b with $f : X \to X$ defined by

$$a \mapsto bba, \quad b \mapsto abb.$$

Then (X, f) is an elementary presentation of a branched solenoid and f does not satisfy the Flattening Axiom. The circle b contains a unique nonbranch fixed point q.

The directed path set $\mathcal{P}_{\{q\}}$ has three loops

$$\alpha = b_2 b_1, \quad \beta = b_2 ab_2, \quad \gamma = b_2 aab_1$$

where b_1 is the arc from the branch point to q and b_2 is from q to the branch point.

Let $X_{\{q\}}$ be a wedge of three circles α, β, γ based at q, and $f_{\{q\}} : X_{\{q\}} \to X_{\{q\}}$ the map induced from $f : X \to X$ by

$$\alpha \mapsto \alpha\beta, \quad \beta \mapsto \alpha\alpha\alpha\gamma, \quad \gamma \mapsto \alpha\alpha\alpha\beta\alpha\gamma.$$

Then $(X_{\{q\}}, f_{\{q\}})$ satisfies the Flattening Axiom. So $(\overline{X}, \overline{f})$ is not topologically conjugate to $(\overline{X}_{\{q\}}, \overline{f}_{\{q\}})$.

5. Canonical SFT covers

In §4, we constructed a new presentation $(X_\mathcal{O}, f_\mathcal{O})$ for the given presentation (X, f) and the finite invariant set $\mathcal{O}$ of X. Associated to this new presentation, we have an SFT $(\Sigma_\mathcal{O}, \sigma_\mathcal{O})$ defined by $(\mathcal{E}_{X_\mathcal{O}}, \tilde{f}_\mathcal{O})$ and a semiconjugacy $p_\mathcal{O} \colon (\Sigma_\mathcal{O}, \sigma_\mathcal{O}) \to (X_\mathcal{O}, f_\mathcal{O})$ constructed by the standard algorithm described in section 2. We will show that this new SFT cover defined by a finite invariant set is canonical. As an application, we show that two solenoids are not conjugate by comparing Bowen-Franks groups of their SFT covers defined by periodic orbits of the same period.

Suppose that (X, f) satisfies all six Axioms, that $\mathcal{O}$ is a finite subset of X such that $f(\mathcal{O}) = \mathcal{O}$, and that $(X_\mathcal{O}, f_\mathcal{O})$ is the presentation defined by (X, f) and $\mathcal{O}$. If $\mathcal{E}_\mathcal{O}$ is the set of edges in $X_\mathcal{O}$, and $\tilde{f}_\mathcal{O} \colon \mathcal{E}_\mathcal{O} \to \mathcal{E}_\mathcal{O}^*$ is the wrapping rule associated to $f_\mathcal{O}$, then $(\mathcal{E}_\mathcal{O}, \tilde{f}_\mathcal{O})$ defines a two-sided SFT $(\Sigma_\mathcal{O}, \sigma_\mathcal{O})$.

There is a well-defined quotient map $p_\mathcal{O} \colon \Sigma_\mathcal{O} \to \overline{X}_\mathcal{O}$ such that $\overline{f}_\mathcal{O} \circ p_\mathcal{O} = p_\mathcal{O} \circ \sigma_\mathcal{O}$. If $\rho_\mathcal{O} \colon X_\mathcal{O} \to X$ is the natural projection which maps each edge e of $X_\mathcal{O}$ to the corresponding path l in X, then $\overline{\rho}_\mathcal{O} \colon \overline{X}_\mathcal{O} \to \overline{X}$ is a conjugacy by Theorem 4.12, and $\overline{\rho}_\mathcal{O} \circ p_\mathcal{O} \colon \Sigma_\mathcal{O} \to \overline{X}$ is a finite-to-one quotient map.

The canonical projection map to the zeroth coordinate $\pi \colon \overline{X} \to X$ induces a bijection $\overline{\mathcal{O}} \leftrightarrow \mathcal{O}$ of finite invariant sets of $\overline{X}$ and X. We will call $(\Sigma_\mathcal{O}, \sigma_\mathcal{O})$ the **SFT cover of $(\overline{X}, \overline{f})$ defined by $\mathcal{O}$** or by $\overline{\mathcal{O}}$.

Example 5.1. Let (X, f) be as in Example 4.8, p the branch point of X which is a fixed point of f, and $\{q, r\}$ a period 2 orbit. Then the 'natural' SFT cover of $(\overline{X}, \overline{f})$ is the SFT $(\Sigma_{\{p\}}, \sigma_{\{p\}})$ defined by the orbit $\{p\}$. From the wrapping rule $\tilde{f} \colon a \mapsto aab, b \mapsto ab$ we see that $(\Sigma_{\{p\}}, \sigma_{\{p\}})$ is represented by the following adjacency matrix.

$$M_{\{p\}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The induced map $\tilde{f}_{\{q,r\}} \colon \mathcal{P}_{\{q,r\}} \to \mathcal{P}_{\{q,r\}}^*$ is given by

$$\alpha \mapsto \gamma\alpha\beta, \quad \beta \mapsto \gamma, \quad \gamma \mapsto \beta\gamma\alpha\beta.$$

So the SFT cover $(\Sigma_{\{q,r\}}, \sigma_{\{q,r\}})$ of $(\overline{X}, \overline{f})$ defined by $\{q, r\}$ is given by the following matrix.

$$M_{\{q,r\}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Remark that $(\overline{X}_{\{p\}}, \overline{f}_{\{p\}})$ is topologically conjugate to $(\overline{X}_{\{q,r\}}, \overline{f}_{\{q,r\}})$ by Theorem 4.12. But $(\Sigma_{\{p\}}, \sigma_{\{p\}})$ is not topologically conjugate to $(\Sigma_{\{q,r\}}, \sigma_{\{q,r\}})$ as the trace of $M_{\{p\}}$ is different from that of $M_{\{q,r\}}$.

Theorem 5.2. *Suppose that (X, f) and (Y, g) satisfy all six Axioms, and that $(\overline{X}, \overline{f})$ is topologically conjugate to $(\overline{Y}, \overline{g})$ by a conjugacy map ϕ. If $\overline{\mathcal{O}}$ is a finite union of periodic orbits of $\overline{f}$ and $\overline{\mathcal{O}'} = \phi(\overline{\mathcal{O}})$, then there is a unique conjugacy*

$\tilde{\phi}\colon (\Sigma_{\mathcal{O}}, \sigma_{\mathcal{O}}) \to (\Sigma_{\mathcal{O}'}, \sigma_{\mathcal{O}'})$ *such that the following diagram commutes.*

$$
\begin{array}{ccc}
(\Sigma_{\mathcal{O}}, \sigma_{\mathcal{O}}) & \xrightarrow{\tilde{\phi}} & (\Sigma_{\mathcal{O}'}, \sigma_{\mathcal{O}'}) \\
p_{\mathcal{O}} \downarrow & & \downarrow p_{\mathcal{O}'} \\
(\overline{X}_{\mathcal{O}}, \overline{f}_{\mathcal{O}}) & & (\overline{Y}_{\mathcal{O}'}, \overline{g}_{\mathcal{O}'}) \\
\overline{p}_{\mathcal{O}} \downarrow & & \downarrow \overline{p}_{\mathcal{O}'} \\
(\overline{X}, \overline{f}) & \xrightarrow{\phi} & (\overline{Y}, \overline{g})
\end{array}
$$

Proof. That ϕ is a conjugacy implies that $\overline{\mathcal{O}'}$ is a finite union of periodic orbits of $\overline{g}$. Let $(X_{\mathcal{O}}, f_{\mathcal{O}})$ and $(Y_{\mathcal{O}'}, g_{\mathcal{O}'})$ be the new graphs with graph maps defined by $\mathcal{O} = \pi(\overline{\mathcal{O}})$ and $\mathcal{O}' = \pi(\overline{\mathcal{O}'})$, respectively. Then by Theorem 4.12, there exist shift equivalence maps $(\rho_{\mathcal{O}}, \psi_{\mathcal{O}})$ for $(X_{\mathcal{O}}, f_{\mathcal{O}})$ and (X, f), and $(\rho_{\mathcal{O}'}, \psi_{\mathcal{O}'})$ for $(Y_{\mathcal{O}'}, g_{\mathcal{O}'})$ and (Y, g). Let $\phi_{\mathcal{O}}\colon \overline{X}_{\mathcal{O}} \to \overline{Y}_{\mathcal{O}'}$ be the conjugacy $(\overline{\rho}_{\mathcal{O}'})^{-1} \circ \phi \circ \overline{\rho}_{\mathcal{O}}$, which lifts ϕ.

By the Ladder Lemma, there is a shift equivalence $r_{\mathcal{O}}\colon X_{\mathcal{O}} \to Y_{\mathcal{O}'}$ and $s_{\mathcal{O}}\colon Y_{\mathcal{O}'} \to X_{\mathcal{O}}$ such that $\overline{r}_{\mathcal{O}} = \phi_{\mathcal{O}}$ and $\overline{s}_{\mathcal{O}} = \phi_{\mathcal{O}}^{-1}$. Since ϕ sends $\overline{\mathcal{O}}$ to $\overline{\mathcal{O}'}$, $r_{\mathcal{O}}$ and $s_{\mathcal{O}}$ are graph maps. Then it follows from Theorem 3.8 that there is a unique conjugacy $\tilde{\phi}\colon (\Sigma_{\mathcal{O}}, \sigma_{\mathcal{O}}) \to (\Sigma_{\mathcal{O}'}, \sigma_{\mathcal{O}'})$ lifting $\phi_{\mathcal{O}}$. Therefore $\tilde{\phi}$ is the unique conjugacy lifting ϕ. $\square$

Remarks 5.3.   (1) It is necessary to assume $\overline{\mathcal{O}'} = \phi(\overline{\mathcal{O}})$. See Example 5.1.
   (2) We need the Flattening Axiom to guarantee that $\phi_{\mathcal{O}}$ is a conjugacy. See Remark 4.13.

Bowen-Franks groups. We say that two dynamical systems (M, ϕ) and (M', ϕ') are **flow equivalent** if they have topologically equivalent suspension flows.

Definition 5.4. ([9, §7]) Let A be an $r \times r$ integral matrix. The **Bowen-Franks group** of A is

$$BF(A) = \operatorname{coker}(Id - A) = \mathbb{Z}^r / \mathbb{Z}^r (Id - A),$$

where $\mathbb{Z}^r (Id - A)$ is the image of $\mathbb{Z}^r$ under the matrix $Id - A$ acting on the right.

If two irreducible SFTs are flow equivalent, then they have isomorphic Bowen-Franks group ([5]).

Example 5.5. Williams posed the following question ([13, 15]): If X is a wedge of two circles a, b, and $g_1, g_2\colon X \to X$ are given by

$$\tilde{g}_1(a) = aabba, \quad \tilde{g}_1(b) = a, \text{ and}$$

$$\tilde{g}_2(a) = ababa, \quad \tilde{g}_2(b) = a,$$

then are g_1 and g_2 shift equivalent?

Ustinov([13]) already showed that they are not shift equivalent to each other using ideas of combinatorial group theory. We will give an alternate argument using the canonical SFT covers. We will compare the Bowen-Franks groups of the SFT covers defined by period-2 orbits. Since conjugacy preserves flow equivalence classes, it suffices to show that there is no bijection between the SFT covers defined by period 2 orbits in (X, g_1) and (X, g_2), respectively, such that the bijection respects the Bowen-Franks groups.

The presentations (X, g_1) and (X, g_2) are given by the following diagrams. So A
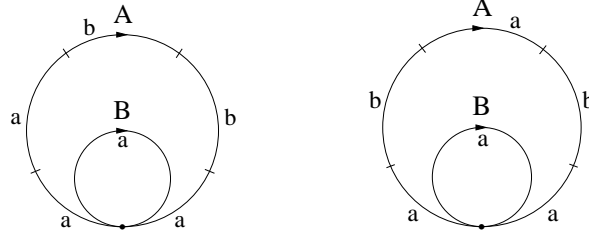
has three subpaths a_1, a_2, a_3 which map to A by g_i, and two subpaths b_1, b_2 which map to B.

Each presentation has five period 2 orbits (excluding fixed points). Let's denote (a_i, a_j) as the period 2 orbit contained in $a_i \cup a_j$ and (b_k, B) as the period 2 orbit in $b_k \cup B$. For (X, g_1), the SFT covers defined by period 2 orbits are represented by the following matrices

$$M(a_1, a_2) = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 1 & 1 \\ 5 & 3 & 1 \end{pmatrix}, \ M(a_1, a_3) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 3 & 4 & 0 \end{pmatrix}, \ M(a_2, a_3) = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \\ 4 & 5 & 0 \end{pmatrix}$$

$$M(b_1, B) = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}, \ \text{and} \ \ M(b_2, B) = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

and for (X, g_2),

$$N(a_1, a_2) = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}, \ N(a_1, a_3) = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \ N(a_2, a_3) = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ 2 & 4 & 1 \end{pmatrix}$$

$$N(b_1, B) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \ \text{and} \ \ N(b_2, B) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

We indicate the computation of $M(a_1, a_2)$ as an example. The points in the periodic orbit (a_1, a_2) and the new graph $X_{(a_1, a_2)}$ are given in the following diagrams. Let p_1 denote the path from the branch point to the point in a_1, p_2 the path from
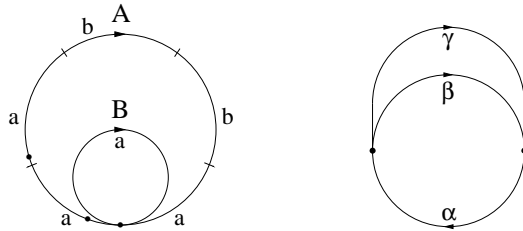
the point in a_1 to the point in a_2, and p_3 the path from the point in a_2 to the branch point. Let $\alpha = p_2$, $\beta = p_3 p_1$, and $\gamma = p_3 b b p_1$. Then the substitution rule $\tilde{g}_{1(a_1, a_2)} \colon \mathcal{E}_{(a_1, a_2)} \to \mathcal{E}^*_{(a_1, a_2)}$ is given by

$$\alpha \mapsto \beta, \quad \beta \mapsto \alpha\gamma\alpha\beta\alpha, \quad \gamma \mapsto \alpha\gamma\alpha\beta\alpha\beta\alpha\beta\alpha.$$

Use the Smith form to compute Bowen-Franks groups ([9, §7.4]) of SFT covers defined by period 2 orbits. Then it is not difficult to obtain that in (X, g_1), $M(a_1, a_3)$ and $M(b_1, B)$ have $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, and $M(a_1, a_2), M(a_2, a_3)$ and $M(b_2, B)$ have $\mathbb{Z}_8$ as their Bowen-Franks groups. And in (X, g_2), $N(a_1, a_2), N(a_1, a_3)$ and $N(a_2, a_3)$ have $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, and $N(b_1, B), N(b_2, B)$ have $\mathbb{Z}_8$. So the number of SFT covers which have the same Bowen-Franks groups are different, and $\overline{g}_1$ is not conjugate to $\overline{g}_2$. Therefore (X, g_1) is not shift equivalent to (X, g_2).

Appendix A. One-sided SFT

Suppose that (X, f) and (Y, g) are presentations of solenoids which are shift equivalent to each other by a shift equivalence $r \colon X \to Y$ and $s \colon Y \to X$. Assume that $\mathcal{O}$ is a finite invariant subset of (X, f) and $\mathcal{O}' = r(\mathcal{O})$, and denote $(\Sigma_{\mathcal{O}}, \sigma_{\mathcal{O}})$ and $(\Sigma_{\mathcal{O}'}, \sigma_{\mathcal{O}'})$ as the SFT covers of $(\overline{X}, \overline{f})$ and $(\overline{Y}, \overline{g})$ defined by $\mathcal{O}$ and $\mathcal{O}'$ respectively. Then $(\Sigma_{\mathcal{O}}, \sigma_{\mathcal{O}})$ and $(\Sigma_{\mathcal{O}'}, \sigma_{\mathcal{O}'})$ are defined by nonnegative integer matrices $M_{\mathcal{O}}$ and $M_{\mathcal{O}'}$, respectively. And we can make **one-sided** SFTs $(S_{\mathcal{O}}, \sigma_{\mathcal{O}})$ and $(S_{\mathcal{O}'}, \sigma_{\mathcal{O}'})$ from $M_{\mathcal{O}}$ and $M_{\mathcal{O}'}$, respectively.

The purpose of this appendix is to give an example in which the one-sided SFTs $(S_{\mathcal{O}}, \sigma_{\mathcal{O}})$ and $(S_{\mathcal{O}'}, \sigma_{\mathcal{O}'})$ are not conjugate.

Example A.1. Let $(Y_{\{p\}}, g_{\{p\}})$ be as in Example 4.10. Let Y_1 be a graph such that $\mathcal{E}_{Y_1} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with a graph map $g_1 \colon Y_1 \to Y_1$ defined by

$$1 \mapsto 1234357, \quad 2, 4, 5, 6 \mapsto 8, \quad 3 \mapsto 167, \quad 7 \mapsto 1, \text{ and } 8 \mapsto 678.$$

The presentation $(Y_{\{p\}}, g_{\{p\}})$ and the graph Y_1 are given in the following diagrams. In Y_1, the fixed point labeled p is the terminal point of the edge 8 and the initial point of the edge 1.
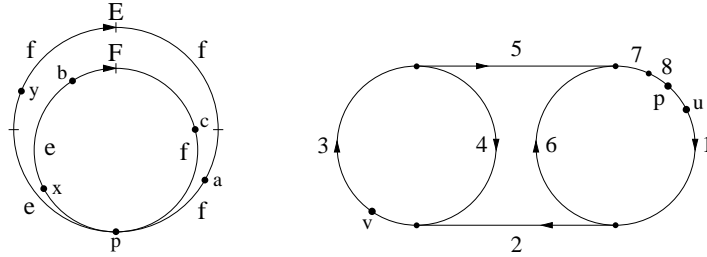
We will define a lag 1 shift equivalence of $g_{\{p\}}$ and g_1 by graph maps r, s under which the points labeled p correspond. The points labeled x, y in $Y_{\{p\}}$ and u, v in Y_1 form period two orbits of $g_{\{p\}}$ and g_1, respectively. The map r will send the points x, y to the points u, v. But, we will see that the covering SFTs associated to $\{x, y\}$ and $\{u, v\}$ are not conjugate as one-sided shifts.

The points a, b in $Y_{\{p\}}$ is a period two orbit, and the point c is the unique point in the edge F such that $g_{\{p\}}(c) = b$. Let e_1 be the path from p to a, e_2 the path from a to p, f_1 the path from p to b, f_2 the path from b to c, and f_3 the path from c to p. Define $r \colon Y_{\{p\}} \to Y_1$ and $s \colon Y_1 \to Y_{\{p\}}$ by

$$\tilde{r} \colon e_1 \mapsto 1234357, \; e_2 \mapsto 8, \; f_1 \mapsto 1, \; f_2 \mapsto 67, \; f_3 \mapsto 8$$
$$\tilde{s} \colon 1 \mapsto e_1, \; 2, 6 \mapsto e_2, \; 3 \mapsto f_1 f_2, \; 4, 5 \mapsto f_3, \; 7 \mapsto f_1, \; 8 \mapsto f_2 f_3.$$

Then $\tilde{r}$ is given by $e \mapsto 12343578$ and $f \mapsto 1678$, and

$$\tilde{s} \circ \tilde{r} \colon e \mapsto e_1 e_2 f_1 f_2 f_3 f_1 f_2 f_3 f_1 f_2 f_3 = efff, \text{ and } f \mapsto e_1 e_2 f_1 f_2 f_3 = ef$$

$$\tilde{r} \circ \tilde{s} \colon 1 \mapsto 1234357, \ 2,4,5,6 \mapsto 8, \ 3 \mapsto 167, \ 7 \mapsto 1, \text{ and } 8 \mapsto 678.$$

Therefore we have $s \circ r = g_{\{p\}}$ and $r \circ s = g_1$, and $(Y_{\{p\}}, g_{\{p\}})$ is shift equivalent to (Y_1, g_1) by r and s.

Now $\mathcal{P}_{\{x,y\}} = \{\alpha, \beta, \gamma\}$ where α is the path from y through p to x, β is the circle F based at x, and γ is the path from x through p to y. And $\mathcal{P}_{\{u,v\}} = \{\delta, \epsilon, \zeta, \eta\}$ where δ is the path from u through the path 2 to v, ϵ is the circle 34 based at v, ζ is the path from v through the path 5 to u, and η is the circle 1678 based at u. The wrapping rules $\tilde{g}_{\{x,y\}} \colon \mathcal{P}_{\{x,y\}} \to \mathcal{P}^*_{\{x,y\}}$ and $\tilde{g}_{\{u,v\}} \colon \mathcal{P}_{\{u,v\}} \to \mathcal{P}^*_{\{u,v\}}$ are given by

$$\tilde{g}_{\{x,y\}} \colon \alpha \mapsto \beta\beta\gamma, \quad \beta \mapsto \alpha\gamma, \quad \gamma \mapsto \alpha\gamma\alpha, \text{ and}$$

$$\tilde{g}_{\{u,v\}} \colon \delta \mapsto \epsilon\zeta, \quad \epsilon \mapsto \eta, \quad \zeta \mapsto \eta\eta\delta, \quad \eta \mapsto \epsilon\zeta\eta\delta.$$

Therefore the SFT covers associated to $\{x,y\}$ and $\{u,v\}$ are presented by the following matrices.

$$M_{\{x,y\}} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \qquad M_{\{u,v\}} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

For

$$R = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

we have $M_{\{x,y\}} = RS$ and $M_{\{u,v\}} = SR$, and the two-sided SFT covers are topologically conjugate (as guaranteed by Theorem 5.2). On the other hand, Williams showed [17] that one-sided SFTs are conjugate if and only if they have the same total column amalgamation matrix; and we see that $M_{\{x,y\}}$ is its own total column amalgamation, whereas the total column amalgamation of $M_{\{u,v\}}$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore these one-sided SFTs are not conjugate.

Appendix B. Relation between Williams' definition and topological definition of 1-solenoids

Let (X, f) be a presentation of a solenoid in our topological sense, that is, assume that it satisfies the Axioms 0-5 of Section 2. In this appendix, we will give X the differentiable structure of a branched 1-manifold, with respect to which f will be an immersion. This will show that our topological 1-solenoids are exactly the systems obtained from Williams' differentiable 1-solenoids by forgetting the differentiable structure.

The precise definition of an n-dimensional branched manifold ([16]) is necessarily somewhat complicated and subtle. In our one-dimensional situation, we will attach to (X, f) a more simple and special structure, from which an interested reader can

easily derive the full immersion of a branched manifold apparatus laid out in [16]. There is some overlap in our ideas and those used by Williams for his Realization Theorem [15, 7.6].

We will cover the graph X with open sets $V_1, \ldots, V_k$. Each V_i will be an interval or a union of intervals intersecting at a vertex. There will be attached maps (*charts*) $\pi_i \colon V_i \to I_i$, where the I_i are open intervals. When V_i is an interval, π_i will be a homeomorphism; in the other case V_i will be a union of intervals the restriction of π_i to each of which is a homeomorphism. Let $V_{ij} = V_i \cap V_j$. Whenever $i \neq j$ and V_{ij} is nonempty, the restriction of the map π_i to V_{ij} will be invertible and the 'change-of-coordinates' map

$$\pi_{i,j} \colon \pi_i(V_{ij}) \to \pi_j(V_{ij}) \text{ given by } x \mapsto \pi_j \circ \pi_i^{-1} x$$

will be invertible and affine. Finally, for each pair i, j such that $f(V_i) \cap V_j$ is nonempty, there will be an invertible affine map $f_{i,j} \colon I_i \to I_j$ such that the restriction of f to $V_i \cap f^{-1}(V_j)$ is a lift of $f_{i,j}$, that is to say, $f_{i,j} \circ \pi(x) = \pi_j \circ f(x)$ for all x in $V_i \cap f^{-1}(V_j)$.

To define the open sets V_i, fix N such that each point of X has a neighborhood whose image under f^N is an interval such that the interior of every interval with endpoints in $f^{-1}(V)$ contains a point in $f^{-N}(V) \setminus f^{-1}(V)$. The sets V_i will be of two sorts. First, the complement of $f^{-N}(V)$ in X will be a disjoint union of open intervals, and each of these will be one of the sets V_i. Second, at each point y in $f^{-N}(V)$, we pick a connected open neighborhood U_y such that the U_y are disjoint; after some shrinking, these will be the remaining V_i. The complement in U_y of y will be the union of a collection $\mathcal{J}(y)$ of disjoint open intervals $J_{y,t}$ (if y is not a vertex, then there are just two); we pass to a shrunken set of V_i for which the images of any pair $J_{y,t}, J_{y',t'}$ are equal or disjoint. This completes the description of the V_i.

To describe the charts, we will use the following result of Williams. Williams proved it for elementary presentations, but his proof works for the non-elementary case and it works with our topological axioms.

Lemma B.1 ([15, 6.2]). *There exist a unique measure μ on X_1 and a unique real number $\lambda > 1$ such that $\mu(X_1) = 1$ and $\mu\big(f_1(I)\big) = \lambda I$ for every small interval contained in an edge.*

First we use this measure μ to define the charts π_i when V_i is an interval: identifying V_i with $(0,1)$, we define $\pi_i(x) = \mu(0, x)$. Next, for each y in $f^{-N}(V)$, define an equivalence relation on $\mathcal{J}(y)$ by declaring two intervals to be equivalent if their images under f^N are equal. This divides each $\mathcal{J}(y)$ into two equivalence classes; making an arbitrary choice, designate one class as $\mathcal{L}(y)$ and the other as $\mathcal{R}(y)$.

Now we can describe the chart π_i for $V_i = U_y$. Identify each $J_{y,t}$ with the interval $(0,1)$, with 0 corresponding to v, and for x in $J_{v,i}$ define

$$\pi_v(x) = \begin{cases} \mu(0,x) & \text{if } J_{v,i} \in R(v) \\ -\mu(0,x) & \text{if } J_{v,i} \in L(v) \ . \end{cases}$$

Finally, define $\pi_i(y) = 0$. This completes the definition of the charts π_i. It is easy to verify that the change-of-coordinate maps $\pi_{i,j}$ are one-to-one and affine as claimed.

It remains to see that f is locally the lift of affine maps as claimed. Suppose that the set $V_{i,j} = V_i \cap f^{-1}(V_j)$ is nonempty. If V_i is not one of the U_y, then by choice of N, $f(V_i) \cap V_j$ is an interval, and by choice of N if $V_j = U_y$, then this interval

is entirely contained in one of the intervals $J(y, t)$; so the required affine map $f_{i,j}$ (with multiplicative constant λ or $-\lambda$) exists. Similarly the required $f_{i,j}$ exists if $V_i = U_y$ and $f(y)$ is not a vertex.

Next, assume $V_i = U_y$ and $f(y)$ is a vertex. For each $J_{y,t}$, let J_t' denote the unique member of $\mathcal{J}(f(y))$ intersecting $f(J_{y,t})$. First suppose that y is not a vertex, so $\mathcal{J}(y) = \{J_{y,1}, J_{y,2}\}$. By the Nonfolding Axiom, the restriction of f^{N+1} to V_i is locally injective at y, and therefore J_1' and J_2' must be in different $\mathcal{L}/\mathcal{R}$ classes. Therefore the required affine map $f_{i,j}$ exists, with multiplicative constant λ or $-\lambda$.

Finally, suppose $V_i = U_y$, y is a vertex and (therefore) $f(y)$ is a vertex. We claim that for any J_1 in $\mathcal{L}(y)$ and J_2 in $\mathcal{R}(y)$, the intervals J_1' and J_2' cannot be in the same equivalence class $\mathcal{L}(f(y))$ or $\mathcal{R}(f(y))$. Given the claim, we can define the required map $f_{i,j}$ as $x \mapsto \lambda x$ (if the $\mathcal{L}, \mathcal{R}$ equivalence class labeling is respected) or $x \mapsto -\lambda x$ (if the labeling is reversed). So it remains to prove the claim.

Suppose the claim is false - suppose there are $J_1 \in \mathcal{L}(y)$ and $J_2 \in \mathcal{R}(y)$ with J_1' and J_2' both in $\mathcal{L}(f(y))$ or both in $\mathcal{R}(f(y))$. Because $f^N(J_1') = f^N(J_2')$ it follows that f^{N+1} is not locally injective at y on the interval which is the union of $\{y\}$, J_1 and J_2. Because the image under f^N of an interval $J_{y,t}$ depends only on the class $\mathcal{L}(y)$ or $\mathcal{R}(y)$ to which it belongs, it follows that the restriction of f^{N+1} to any open interval containing y is not locally one-to-one. Now pick a positive integer k and a point w which is not a vertex such that $f^k(w) = y$. Then f^{k+N+1} is not locally injective at w. This contradicts the Nonfolding Axiom, and concludes the proof. We record the result as the following Proposition.

Proposition B.2. *Suppose (X, f) is a presentation of a topologically defined 1-solenoid. Then X can be given a differentiable structure making it a branched manifold on which f is an immersion presenting a Williams solenoid.*

Example B.3. Let X be given by the following graph, and $f : X \to X$ is given by
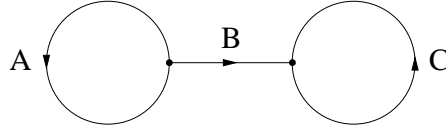
$a \mapsto b^{-1}ab, \ b \mapsto cb^{-1}a, \ c \mapsto bcb^{-1}$.

If we redraw X as the following graph, then X is a branched manifold, and (X, f) is a presentation in the sense of Williams.
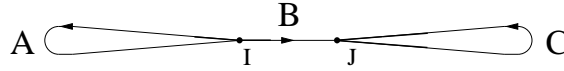
In contrast to the solenoid case, the following example shows that the set of Williams' branched solenoids is a proper subset of the topologically defined branched solenoids.

Example B.4. Let X be as in the previous example, and $g : X \to X$ is given by

$$a \mapsto cb^{-1}ab, \ b \mapsto c^{-1}b^{-1}, \ c \mapsto a.$$

Then (X, g) satisfies all Axioms except for the Flattening Axiom. So (X, g) is a presentation of a branched solenoid according to the topological definition.

At each branch point of X, there are three choices of differentiable structure, that is, three arcs are parallel to each other or two arcs are parallel and the other arc is 180 degrees to these two arcs. The second graph in Example B.3 and Figure B.4 show three possible differentiable structures at the left branch point when the circle c is fixed. And similar differentiable structures can be given to the right branch point.

In each choice of differential structures at both branched points, it is not difficult to find a smooth interval which is mapped to a non-smooth interval homeomorphically by $g\colon X \to X$. For example, if X is the second graph in Example B.3, then the interval I is mapped to the interval J by g homeomorphically. Hence (X, g) cannot be a presentation of a branched solenoid according to Williams' definition.
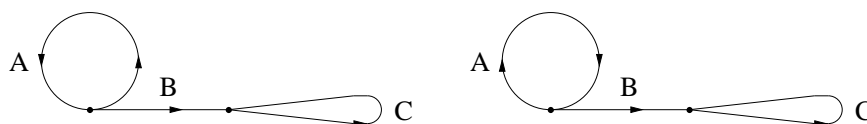

FIGURE 4. Other differentiable structures at the branch point of A when the circle c is fixed.

REFERENCES

1. R. Adler and B. Marcus, *Topological entropy and equivalence of dynamical systems* Mem. Amer. Math. Soc. **219**, 1979.
2. J. Anderson and I. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras* Ergodic Th. and Dynam. Sys. **18** (1998), 509–37.
3. M. Barge, J. Jacklitch, and G. Vago, *Homeomorphisms of One-dimensional Inverse Limits with Applications to Substitution Tilings, Unstable Manifolds, and Tent Maps*, Contemp. Math., to appear.
4. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math. **470**, Springer-Verlag, 1975.
5. R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets*, Ann. of Math. **106**, (1977), 73-92.
6. F. Durand, B. Host and C. Skau, *Substitution dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Th. and Dynam. Sys., to appear.
7. D. Fried, *Finitely presented dynamical systems*, Ergodic Th. and Dynamical Sys. **7** (1987), 489-507.
8. W. Krieger, *On sofic systems I*, Israel J. Math. **48** (1984), 305-330.
9. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, 1995.
10. C. Robinson, *Dynamical systems: stability, symbolic dynamics, and chaos* (2nd Edition), CRC Press, 1999.
11. M. Snavely, *Markov partitions for the two-dimensional torus*, Proc. Amer. Math. Soc. **113** (1991), 517-527.
12. R. Swanson and H. Volkmer, *Invariants of weak equivalence in primitive matrices*, Ergodic Th. and Dynam. Sys., to appear.
13. Y. Ustinov, *Algebraic invariants of topological conjugacy of solenoids*, Mat. Zametki **42** (1987), 132-144.
14. R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473-487.
15. R. F. Williams, *Classification of 1-dimensional attractors*, Proc. Symp. Pure Math. **14** (1970), 341-361.

16. R. F. Williams, *Expanding attractors*, IHES Publ. Math. **43** (1974), 169-203.
17. R. F. Williams, *Classification of subshifts of finite type*, Ann. of Math. (2) **98** (1973), 120-153.
18. I. Yi *Ordered group invariants for one-dimensional spaces*, preprint, 1999.

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